Commutative Geometries are Spin Manifolds

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Abstract

In [1], Connes presented axioms governing noncommutative geometry. He went on to claim that when specialised to the commutative case, these axioms recover spin or spin^c geometry depending on whether the geometry is "real" or not. We attempt to flesh out the details of Connes' ideas. As an illustration we present a proof of his claim, partly extending the validity of the result to pseudo-Riemannian spin manifolds. Throughout we are as explicit and elementary as possible.

1 Introduction

The usual description of noncommutative geometry takes as its basic data unbounded Fredholm modules, known as K-cycles or spectral triples. These are triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is an involutive algebra represented on the Hilbert space $\mathcal{H}$. The operator $\mathcal{D}$ is a closed, unbounded operator on $\mathcal{H}$ with compact resolvent such that the commutator $[\mathcal{D}, \pi(a)]$ is a bounded operator for every $a \in \mathcal{A}$. Here π is the representation of $\mathcal{A}$ in $\mathcal{H}$. We also suppose that we are given an integer p called the degree of summability which governs the dimension of the geometry. If p is even, the Hilbert space is $\mathbf{Z}_2$-graded in such a way that the operator $\mathcal{D}$ is odd.

In [1], axioms were set down for noncommutative geometry. It is in this framework that Connes states his theorem recovering spin manifolds from commutative geometries. Perhaps the most important aspect of this theorem is that it provides sufficient conditions for the spectrum of a C^*-algebra to be a (spin^c) manifold. It also gives credence to the idea that spectral triples obeying the axioms should be regarded as noncommutative manifolds.

Let us briefly describe the central portion of the proof. Showing that the spectrum of $\mathcal{A}$ is actually a manifold relies on the interplay of several abstract structures and the axioms controlling their representation. At the abstract level we can define the universal differential algebra of $\mathcal{A}$, denoted $\Omega^*(\mathcal{A})$. The underlying linear space of $\Omega^*(\mathcal{A})$ is isomorphic to the chain complex from which we construct the Hochschild homology of $\mathcal{A}$. In the commutative case, there is also another definition of the differential forms over $\mathcal{A}$. This algebra, $\widehat{\Omega}^*(\mathcal{A})$, is skew-commutative and when the algebra $\mathcal{A}$ is 'smooth', [2], it coincides with the Hochschild homology of $\mathcal{A}$. For a commutative algebra it is always the case that Hochschild homology contains $\widehat{\Omega}^*(\mathcal{A})$ as a direct summand.

The axioms, among other things, ensure that we end up with a faithful representation of $\widehat{\Omega}^*(\mathcal{A})$. The process begins by constructing a representation of $\Omega^*(\mathcal{A})$ from a representation π of $\mathcal{A}$, which we may assume is faithful. This is done using the operator $\mathcal{D}$ introduced above by setting

$$\pi(\delta a) = [\mathcal{D}, \pi(a)] \quad \forall a \in \mathcal{A}, \tag{1}$$

where $\Omega^*(\mathcal{A})$ is generated by the symbols δa, $a \in \mathcal{A}$. There are three axioms/assumptions controlling this representation. The first is Connes' first order condition. This demands that $[[\mathcal{D}, \pi(a)], \pi(b)] = 0$ for all $a, b \in \mathcal{A}$, at least in the commutative case. It turns out that the kernel of a representation of $\Omega^*(\mathcal{A})$ obeying this condition is precisely the image of the Hochschild boundary. Thus our representation descends to a representation of Hochschild homology $HH_*(\mathcal{A}) \subseteq \Omega^*(\mathcal{A})$, and is moreover faithful.

The algebra $\Omega^*_{\mathcal{D}}(\mathcal{A}) := \pi(\Omega^*(\mathcal{A}))$ is no longer a differential algebra. To remedy this, one quotients out the 'junk' forms, and these turn out to be the submodule generated over $\mathcal{A}$ by graded commutators and the image of the Hochschild boundary. Thus the algebra, $\Lambda^*_{\mathcal{D}}(\mathcal{A})$, that we arrive at after removing the junk is skew-commutative, and we will show that it is isomorphic to $\widehat{\Omega}^*(\mathcal{A})$. We will then prove that the representation of Hochschild homology with values in $\Lambda^*_{\mathcal{D}}(\mathcal{A})$ is still faithful, showing that $\Lambda^*_{\mathcal{D}}(\mathcal{A}) \cong \widehat{\Omega}^*(\mathcal{A}) \cong HH_*(\mathcal{A})$. This is a necessary, though not sufficient, condition for the algebra $\mathcal{A}$ to be smooth. Note that by virtue of the first order condition, both $\Omega^*_{\mathcal{D}}(\mathcal{A})$ and $\Lambda^*_{\mathcal{D}}(\mathcal{A})$ are symmetric $\mathcal{A}$ bimodules, and so may be considered to be (left or right) modules over $\mathcal{A} \otimes \mathcal{A}$.

Returning to the axioms, the critical assumption to show that the spectrum is indeed a manifold is the existence of a Hochschild p-cycle which is represented by 1 or the $\mathbf{Z}_2$-grading depending on whether p is odd or even respectively. The main consequence of this is that this cycle is nowhere vanishing as a section of $\pi(\Omega^p(\mathcal{A}))$. As it is a cycle, we know that it lies in the skew-commutative part of the algebra, and this will allow us to find generators of the differential algebra over $\mathcal{A}$ and construct coordinate charts on the spectrum. Indeed, the non-vanishing of this cycle is the most stringent axiom, as it enforces the underlying p-dimensionality of the spectrum.

Thus we have a two step reduction process

$$\Omega^*(\mathcal{A}) \to \Omega^*_{\mathcal{D}}(\mathcal{A}) \to \Lambda^*_{\mathcal{D}}(\mathcal{A}), \tag{2}$$

and the third axiom referred to above is needed to control the behaviour of the intermediate algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$, as well as to ensure that our algebra is indeed smooth. Recalling that $\mathcal{D}$ is required to be closed and self-adjoint, we demand that for all $a \in \mathcal{A}$

$$\delta^n(\pi(a)), \quad \delta^n([\mathcal{D}, \pi(a)]) \text{ are bounded for all } n,\ \delta(x) = [|\mathcal{D}|, x]. \tag{3}$$

It is easy to imagine that this could be used to formulate smoothness conditions, but it also forces $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be a (possibly twisted) representation of the complexified Clifford algebra of the cotangent bundle of the spectrum. As the representation is assumed to be irreducible, we will have shown that the spectrum is a spin^c manifold. Inclusion of the reality axiom will then show that the spectrum is actually spin. The detailed description of these matters requires a great deal of work.

We begin in Section 2 with some more or less standard background results. These will be required in Section 3, where we present our basic definitions and the axioms, as well as in Section 4 where we state and prove Connes' result. Section 5 addresses the issue of abstract characterisation of algebras having "geometric representations."

2 Background

Although there are now several good introductory accounts of noncommutative geometry, eg [3], to make this paper as self-contained as possible, we will quote a number of results necessary for the proof of Connes' result and the analysis of the axioms.

2.1 Pointset Topology

The point set topology of a compact Hausdorff space X is completely encoded by the C^*-algebra of continuous functions on X, $C(X)$. This is captured in the Gel'fand-Naimark Theorem.

Theorem 1 *For every commutative C^*-algebra A, there exists a Hausdorff space X such that $A \cong C_0(X)$. If A is unital, then X is compact.*

In the above, $C_0(X)$ means the continuous functions on X which tend to zero at infinity. In the compact case this reduces to $C(X)$. We can describe X explicitly as $X = Spec(A) = \{$maximal ideals of $A\} = \{$pure states of $A\}=\{$unitary equivalence classes of irreducible representations$\}$. The weak* topology on the pure state space is what gives us compactness, and translates into the topology of pointwise convergence for the states. While this theorem provides much of the motivation for the use of C^*-algebras in the context of "noncommutative topology," much of their utility comes from the other Gel'fand-Naimark theorem.

Theorem 2 *Every C^*-algebra admits a faithful and isometric representation as a norm closed self-adjoint *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

In the classical (commutative) case there are a number of results showing that we really can recover all information about the space X from the algebra of continuous functions $C(X)$. For instance, closed sets correspond to norm closed ideals, and so single points to maximal ideals. The latter statement is proved using the correspondence pure state$\leftrightarrow$kernel of pure state; this is the way that the Gel'fand-Naimark theorem is proved, and is not valid in the noncommutative case, [3]. There are many other such correspondences, see [4], not all of which still make sense in the noncommutative case, for the simple reason that the three descriptions of the spectrum no longer coincide for a general C^*-algebra, [3]. On this cautionary note, let us turn to the most important correspondence; the Serre-Swan theorem, [5].

Theorem 3 *Let X be a compact Hausdorff space. Then a $C(X)$-module V is isomorphic (as a module) to a module $\Gamma(X, E)$ of continuous sections of a complex vector bundle $E \to X$ if and only if V is finitely generated and projective.*

We abbreviate finitely generated and projective to the now common phrase finite projective. This is equivalent to the following. If V is a finite projective A-module, then there is an idempotent $e^2 = e \in M_N(A)$, the $N \times N$ matrix algebra over A, for some N such that $V \cong eA^N$. Thus V is a direct summand of a free module. We would like then to treat finite projective C^*-modules as noncommutative generalisations of vector bundles. Ideally we would like the idempotent e to be a projection, i.e. self-adjoint. Since every complex vector bundle admits an Hermitian structure, this is easy to formulate in the commutative case. In the general case we define an Hermitian structure on a right A-module V to be a sesquilinear map $\langle \cdot, \cdot \rangle : V \times V \to A$ such that $\forall a, b \in A, v, w \in V$

1) $\langle av, bw \rangle = a^* \langle v, w \rangle b$,
2) $\langle v, w \rangle = \langle w, v \rangle^*$,
3) $\langle v, v \rangle \geq 0, \quad \langle v, v \rangle = 0 \Rightarrow v = 0$.

A finite projective module always admits Hermitian structures (and connections, for those looking ahead). Such an Hermitian structure is said to be nondegenerate if it gives an isomorphism onto the dual module. This corresponds to the usual notion of nondegeneracy in the classical case, [3]. We also have the following.

Theorem 4 *Let A be a C^*-algebra. If V is a finite projective A-module with a nondegenerate Hermitian structure, then $V \cong eA^N$ for some idempotent $e \in M_N(A)$ and furthermore $e = e^*$, where * is the composition of matrix transposition and the * in A.*

We shall regard finite projective modules over a C^*-algebra as the noncommutative version of (the sections of) vector bundles over a noncommutative space. As a last point before moving on, we note that for every bundle on a smooth manifold, there is an essentially unique smooth bundle. For more information on all these results, see [3, 6].

2.2 Algebraic Topology

Bundle theory leads us quite naturally to K-theory in the commutative case, and to further demonstrate the utility of defining noncommutative bundles to be finite projective modules, we find that this definition allows us to extend K-theory to the noncommutative domain as well. Moreover, in the commutative case (at least for finite simplicial complexes), K-theory recovers the ordinary cohomology via the Chern character, and the usual cohomology theories make no sense whatsoever for a noncommutative space. Thus K-theory becomes the cohomological tool of choice in the noncommutative setting.

This is not the whole story, however, for we know that in the smooth case we can formulate ordinary cohomology in geometric terms via differential forms. Noncommutatively speaking, differential forms correspond to the Hochschild homology of the algebra of smooth functions on the space, whereas the de Rham cohomology is obtained by considering the closely related theory, cyclic homology, [6]. The latter, or more properly the periodic version of the theory, see [2, 6], is the proper receptacle for the Chern character (in both homology and cohomology).

We can define K-theory for any C^*-algebra and K-homology for a class of C^*-algebras and certain of their dense subalgebras. For any of these algebras A, elements of $K^*(A)$ may be regarded as equivalence classes of Fredholm modules $[(\mathcal{H}, F, \Gamma)]$. These consist of a representation $\pi : A \to \mathcal{B}(\mathcal{H})$, an operator $F : \mathcal{H} \to \mathcal{H}$ such that $F = F^*$, $F^2 = 1$, and $[F, \pi(a)]$ is compact for all $a \in A$. If $\Gamma = 1$, the class defined by the module is said to be odd, and it resides in $K^1(A)$. If $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $[\Gamma, \pi(a)] = 0$ for all $a \in A$ and $\Gamma F + F\Gamma = 0$, then we call the class even, and it resides in $K^0(A)$. For complex algebras, Bott periodicity says that these are essentially the only K-homology groups of A. For the case of a commutative C^*-algebra, $C_0(X)$, this coincides with the (analytic) K-homology of X. A relative group can be defined as well, along with a reduced group for dealing with locally compact spaces (non-unital algebras), [7, 8, 9, 10].

The K-theory of the algebra A may be described as equivalence classes of idempotents, $K^0(A)$, and unitaries, $K^1(A)$, in $M_\infty(A)$ and $GL_\infty(A)$ respectively. Again, relative and reduced groups can be defined. We will find it useful when discussing the cap product to denote elements corresponding to actual idempotents or unitaries by $[(e, N)]$, or $[(u, N)]$ respectively, with $e, u \in M_N(A)$. Much of what follows could be translated into KK-theory or E-theory, but we shall be content with the simple presentation below.

The duality pairing between K-theory and K-homology can be broken into two steps. First one uses the cap product, described below, and then acts on the resulting K-homology class with the natural index map. This map, $Index : K^0(A) \to K^0(\mathbf{C}) = \mathbf{Z}$ is given by

$$Index([(\mathcal{H}, F, \Gamma)]) = Index(\frac{1-\Gamma}{2} F \frac{1+\Gamma}{2}). \tag{4}$$

On the right we mean the usual index of Fredholm operators, and the beauty of the Fredholm module formulation is that the $Index$ map is well-defined. The $Index$ map can also be defined on $K^1(A)$, but as the operators in here are all self-adjoint, it always gives zero. For this reason we will avoid mention of the "odd" product rules for the cap product.

The cap product, $\cap$, in K-theory will allow us to formulate Poincaré duality in due course. It is a map

$$\cap : K_*(A) \times K^*(A) \to K^*(A) \tag{5}$$

with the even valued terms given on simple elements by the following rules:

$$K_0(A) \times K^0(A) \to K^0(A)$$
$$[(e,N)] \cap [(\mathcal{H},F,\Gamma)] = [(e\mathcal{H}_+^N \oplus e\mathcal{H}_-^N, \begin{pmatrix} 0 & e\tilde{F}^* \otimes 1_N e \\ e\tilde{F} \otimes 1_N e & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix})], \tag{6}$$

where $\mathcal{H}_\pm = \frac{1 \pm \Gamma}{2}\mathcal{H}$, and $F = \begin{pmatrix} 0 & \tilde{F}^* \\ \tilde{F} & 0 \end{pmatrix}$. As every idempotent determines a finite projective module, this is easily seen to be just twisting the Fredholm module by the module given by $[(e,N)]$. The product of a unitary and an odd Fredholm module will lead to the odd index,

$$K_1(A) \times K^1(A) \to K^0(A)$$
$$[(u,N)] \cap [(\mathcal{H},F)] = [((\tfrac{1+F}{2}\mathcal{H})^{2N}, \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}, \begin{pmatrix} \frac{1+F}{2} \otimes 1_N & 0 \\ 0 & -\frac{1+F}{2} \otimes 1_N \end{pmatrix})]. \tag{7}$$

The cap product turns $K^*(A)$ into a module over $K_*(A)$. Usually, one is given a Fredholm module, $\mu = [(\mathcal{H},F,\Gamma)]$, where $\Gamma = 1$ if the module is odd, and then considers the $Index$ as a map on $K_*(A)$ via $Index(k) := Index(k \cap \mu)$, for any $k \in K_*(A)$.

Let us for a minute suppose that A is commutative. If $X = Spec(A)$ is a compact finite simplicial complex, there are isomorphisms

$$K^*(X) \otimes \mathbf{Q} \cong K_*(A) \otimes \mathbf{Q} \stackrel{ch^*}{\to} H^*(X,\mathbf{Q}) \tag{8}$$

and

$$K_*(X) \otimes \mathbf{Q} \cong K^*(A) \otimes \mathbf{Q} \stackrel{ch_*}{\to} H_*(X,\mathbf{Q}) \tag{9}$$

given by the Chern characters. Here H^* and H_* are the ordinary (co)homology of X. Note that a (co)homology theory for spaces is a homology(co) theory for algebras.

Important for us is that these isomorphisms also preserve the cap product, both in K-theory and ordinary (co)homology, so that the following diagram

$$\begin{array}{ccc} K^*(X) \otimes \mathbf{Q} & \stackrel{\cap\mu}{\to} & K_*(X) \otimes \mathbf{Q} \\ ch^* \downarrow & & ch_* \downarrow \\ H^*(X,\mathbf{Q}) & \stackrel{\cap ch_*(\mu)}{\to} & H_*(X,\mathbf{Q}) \end{array}$$

commutes for any $\mu \in K_*(X)$. So if X is a finite simplicial complex satisfying Poincaré Duality in K-theory, that is there exists $\mu \in K^*(C(X))$ such that $\cap\mu : K_*(C(X)) \to K^*(C(X))$ is an isomorphism, then there exists $[X] = ch_*(\mu) \in H_*(X,\mathbf{Q})$ such that

$$\cap [X] : H^*(X,\mathbf{Q}) \to H_*(X,\mathbf{Q}) \tag{10}$$

is an isomorphism. If $ch_*(\mu) \in H_p(X, \mathbf{Q})$ for some p, then we would know that X satisfied Poincaré duality in ordinary (co)homology, which is certainly a necessary condition for X to be a manifold.

We note in passing that K-theory has only even and odd components, whereas the usual (co)homology is graded by $\mathbf{Z}$. This is not such a problem if we replace (co)homology with periodic cyclic homology(co). In the case of a classical manifold it gives the same results as the usual theory, but it is naturally $\mathbf{Z}_2$-graded. Though we do not want to discuss cyclic (co)homology in this paper, we note that the appropriate replacement for the commuting square above in the noncommutative case is the following.

$$\begin{array}{ccc} K_*(A) \otimes \mathbf{Q} & \stackrel{\cap\mu}{\to} & K^*(A) \otimes \mathbf{Q} \\ ch_* \downarrow & & ch^* \downarrow \\ H_*^{per}(A) \otimes \mathbf{Q} & \stackrel{\cap ch^*(\mu)}{\to} & H^*_{per}(A) \otimes \mathbf{Q} \end{array}$$

Moreover, the periodic theory is the natural receptacle for the Chern character in the not necessarily commutative case, provided that A is an algebra over a field containing $\mathbf{Q}$. For more information on Poincaré duality in noncommutative geometry, including details of the induced maps on the various homology groups, see [6, 11].

2.3 Measure

On the analytical front we have to relate the noncommutative integral given by the Dixmier trace to the usual measure theoretic tools. This is achieved using two results of Connes; one building on the work of Wodzicki, [12], and the other on the work of Voiculescu, [13]. For more detailed information on these results, see [6] and [12, 13].

To define the Dixmier trace and relate it to Lebesgue measure, we require the definitions of several normed ideals of compact operators on Hilbert space. The first of these is

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{N} \mu_n(T) = O(\log N)\} \tag{11}$$

with norm

$$\| T \|_{1,\infty} = \sup_{N\geq 2} \frac{1}{\log N} \sum_{n=0}^{N} \mu_n(T). \tag{12}$$

In the above the $\mu_n(T)$ are the eigenvalues of $|T| = \sqrt{TT^*}$ arranged in decreasing order and repeated according to multiplicity so that $\mu_0(T) \geq \mu_1(T) \geq \ldots$. This ideal will be the domain of definition of the Dixmier trace. Related to this ideal are the ideals $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ for $1 < p < \infty$ defined as follows;

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{N} \mu_n(T) = O(N^{1-\frac{1}{p}})\} \tag{13}$$

with norm

$$\| T \|_{p,\infty} = \sup_{N\geq 1} \frac{1}{N^{1-\frac{1}{p}}} \sum_{n=0}^{N} \mu_n(T). \tag{14}$$

We introduce these ideals because if $T_i \in \mathcal{L}^{(p_i,\infty)}(\mathcal{H})$ for $i = 1, \ldots, n$ and $\sum \frac{1}{p_i} = 1$, then the product $T_1 \cdots T_n \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. In particular, if $T \in \mathcal{L}^{(n,\infty)}(\mathcal{H})$ then $T^n \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

We want to define the Dixmier trace so that it returns the coefficient of the logarithmically divergent part of the trace of an operator. Unfortunately, since $(1/\log N)\sum^N \mu_n(T)$ is in general only a bounded sequence, we can not take the limit in a well-defined way. The Dixmier trace is usually defined in terms of linear functionals on bounded sequences satisfying certain properties. One of these properties is that if the above sequence is convergent, the linear functional returns the limit. In this case, the result is independent of which linear functional is used. So, for $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ with $T \geq 0$, we say that T is measurable if

$$-\!\!\!\!\!\!\int T := \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N} \mu_n(T) \tag{15}$$

exists. Moreover, $-\!\!\!\!\!\int$ is linear on measurable operators, and we extend it by linearity to not necessarily positive operators. Then $-\!\!\!\!\!\int$ satisfies the following properties, [6]:

1)The space of measurable operators is a closed (in the $(1,\infty)$ norm) linear space invariant under conjugation by invertible bounded operators and contains $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$, the closure of the finite rank operators in the $(1,\infty)$ norm;

2) If $T \geq 0$ then $-\!\!\!\!\!\int T \geq 0$;

3) For all $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ with T measurable, we have $-\!\!\!\!\!\int TS = -\!\!\!\!\!\int ST$;

4) $-\!\!\!\!\!\int$ depends only on $\mathcal{H}$ as a topological vector space;

5) $-\!\!\!\!\!\int$ vanishes on $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$.

In the case that $\mathcal{A}$ is finite dimensional and represented on a finite dimensional Hilbert space, all these ideals of compact operators coincide, and the Dixmier trace reduces to the ordinary trace, [6]. Next we relate this operator theoretic definition to geometry.

If P is a pseudo differential operator acting on sections of a vector bundle $E \to M$ over a manifold M of dimension p, it has a symbol $\sigma(P)$. The Wodzicki residue of P is defined by

$$WRes(P) = \frac{1}{p(2\pi)^p} \int_{S^*M} trace_E \sigma_{-p}(P)(x,\xi)\sqrt{g}dxd\xi. \tag{16}$$

In the above S^*M is the cosphere bundle with respect to some metric g, and $\sigma_{-p}(P)$ is the part of the symbol of P homogenous of order $-p$. In particular, if P is of order strictly less than $-p$, $WRes(P) = 0$. The interesting thing about the Wodzicki residue is that although symbols other than principal symbols are coordinate dependent, the Wodzicki residue depends only on the conformal class of the metric [6]. It is also a trace on the algebra of pseudodifferential operators, and we have the following result from Connes, [14, 6].

Theorem 5 *Let T be a pseudodifferential operator of order $-p$ acting on sections of a smooth bundle $E \to M$ on a p dimensional manifold M. Then as an operator on $\mathcal{H} = L^2(M,E)$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, T is measurable and $-\!\!\!\!\!\int T = WRes(T)$.*

It can also be shown that the Wodzicki residue is the unique trace on pseudodifferential operators extending the Dixmier trace, [14]. Hence we can make sense of $-\!\!\!\!\!\int T$ for any pseudodifferential operator on a manifold by using the Wodzicki residue. This is done by setting $-\!\!\!\!\!\int T = WRes(T)$. In particular, if T is of order strictly less than $-p = -\dim M$, then $-\!\!\!\!\!\int T = 0$. This will be important for us later in relation to gravity actions. Before moving on, we note that when we are dealing with the noncommutative case there is an extended notion of pseudodifferential operators, symbols and Wodzicki residue which reduces to the usual notion in the commutative case; see [15].

The other connection of the Dixmier trace to our work is its relation to the Lebesgue measure.

Since the Dixmier trace acts on operators on Hilbert space we might expect it to be related to measure theory via the spectral theorem. Indeed this is true, but we must backtrack a little into perturbation theory.

The Kato-Rosenblum theorem, [16], states that for a self-adjoint operator T on Hilbert space, the absolutely continuous part of T is (up to unitary equivalence) invariant under trace class perturbation. This result does not extend to the joint absolutely continuous spectrum of more than one operator. Voiculescu shows that for a p-tuple of commuting self-adjoint operators $(T_1, \ldots, T_p)$, the absolutely continuous part of their joint spectrum is (up to unitary equivalence) invariant under perturbation by a p-tuple of operators $(A_1, \ldots, A_p)$ with $A_i \in \mathcal{L}^{(p,1)}(\mathcal{H})$. This ideal is given by

$$\mathcal{L}^{(p,1)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sum_{n=0}^{\infty} n^{\frac{1}{p}-1}\mu_n(T) < \infty\}, \tag{17}$$

with norm given by the above sum.

Voiculescu, [13], was lead to investigate, for X a finite subset of $\mathcal{B}(\mathcal{H})$ and J a normed ideal of compact operators, the obstruction to finding an approximate unit quasi-central relative to X. That is, an approximate unit whose commutators with elements of X all lie in J. To do this, he introduced the following measure of this obstruction

$$k_J(X) = \liminf_{A \in R_1^+, A \to 1} \| [A, X] \|_J . \tag{18}$$

Here R_1^+ is the unit interval $0 \leq A \leq 1$ in the finite rank operators, and in terms of the norm $\| \cdot \|_J$ on J, $\| [A, X] \|_J = \sup_{T \in X} \| [A, T] \|_J$. With this tool in hand Voiculescu proves the following result.

Theorem 6 *Let $T_1, \ldots, T_p$ be commuting self-adjoint operators on the Hilbert space $\mathcal{H}$ and $E_{ac} \subset \mathbf{R}^p$ be the absolutely continuous part of their joint spectrum. Then if the multiplicity function $m(x)$ is integrable, we have*

$$\gamma_p \int_{E_{ac}} m(x) d^p x = (k_{\mathcal{L}^{(p,1)}}(\{T_1, \ldots, T_p\}))^p \tag{19}$$

where $\gamma_p \in (0, \infty)$ is a constant.

This result seems a little out of place, as we are using $\mathcal{L}^{(1,\infty)}$ as our measurable operators. However, Connes proves the following, [6, pp 311-313].

Theorem 7 *Let D be a self-adjoint, invertible, unbounded operator on the Hilbert space $\mathcal{H}$, and let $p \in (1, \infty)$. Then for any set $X \subset \mathcal{B}(\mathcal{H})$ we have*

$$k_{\mathcal{L}^{(p,1)}}(X) \leq C_p (\sup_{T \in X} \| [D, T] \|) (⨍ |D|^{-p})^{1/p}, \tag{20}$$

where C_p is a constant.

The case $p = 1$ must be handled separately. In this paper we will be dealing only with compact manifolds, and so in dimension 1, we have only the circle. We will check this case explicitly in the body of the proof. So ignoring dimension 1 for now, we have the following, [6].

Theorem 8 *Let p and D be as above, with $D^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ and suppose that $\mathcal{A}$ is an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ such that $[D,a]$ is bounded for all $a \in \mathcal{A}$. Then*

1) Setting $\tau(a) = ⨍ a|D|^{-p}$ defines a trace on $\mathcal{A}$. This trace is nonzero if $k_{\mathcal{L}^{(p,1)}}(\mathcal{A}) \neq 0$.

2) Let p be an integer and $a_1, \ldots, a_p \in \mathcal{A}$ commuting self-adjoint elements. Then the absolutely continuous part of their spectral measure

$$\mu_{ac}(f) = \int_{E_{ac}} f(x)m(x)d^px \tag{21}$$

is absolutely continuous with respect to the measure

$$\tau(f) = \tau(f(a_1, \ldots, a_p)) = ⨍ f|D|^{-p}, \quad \forall f \in C_c^\infty(\mathbf{R}^p). \tag{22}$$

Combining the results on the Wodzicki residue and these last results of Voiculescu and Connes, we will be able to show that the measure on a commutative geometry is a constant multiple of the measure defined in the usual way.

The hypothesis of invertibility used in the above theorems for the operator D can be removed provided $\ker D$ is finite dimensional. Then we can add to D a finite rank operator in order to obtain an invertible operator, and the Dixmier trace will be unchanged. For these purely measure theoretic purposes, simply taking $D^{-1} = 0$ on $\ker D$ is fine. More care must be taken with $\ker D$ in the definition of the associated Fredholm module; see [6].

2.4 Geometry

We will avoid discussion of cyclic (co)homology in this paper, as we do not absolutely require it. More important for this paper, is the universal differential algebra construction, and its relation to Hochschild homology.

The (reduced) Hochschild homology of an algebra $\mathcal{A}$ with coefficients in a bimodule M is defined in terms of the chain complex $C_n(M) = M \otimes \tilde{\mathcal{A}}^{\otimes n}$ with boundary map $b : C_n(M) \to C_{n-1}(M)$

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_n) = & \, ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned} \tag{23}$$

Here $\tilde{\mathcal{A}} = \mathcal{A}/\mathbf{C}$. In the event that $M = \mathcal{A}$, we denote $C_n(M) := C_n(\mathcal{A})$ and the resulting homology by $HH_*(\mathcal{A})$, otherwise by $HH_*(\mathcal{A}, M)$. Though we will be concerned with the commutative case, the general setting uses $M = \mathcal{A} \otimes \mathcal{A}^{op}$ with bimodule structure $a(b \otimes c^{op})d = abd \otimes c^{op}$, with $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$. With this structure it is clear that

$$HH_*(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op}) \cong \mathcal{A} \otimes \mathcal{A}^{op} \otimes HH_*(\mathcal{A}) \cong \mathcal{A} \otimes HH_*(\mathcal{A}) \otimes \mathcal{A}^{op}. \tag{24}$$

We will also require topological Hochschild homology. Suppose we have an algebra $\mathcal{A}$ which is endowed with a locally convex and Hausdorff topology such that $\mathcal{A}$ is complete. This is equivalent to requiring that for any continuous semi-norm p on $\mathcal{A}$ there are continuous semi-norms p', q' such that $p(ab) \leq p'(a)q'(b)$ for all $a, b \in \mathcal{A}$. In particular, the product is (separately) continuous. Any algebra with a topology given by an infinite family of semi-norms

in such a way that the underlying linear space is a Frechet space satisfies this property, and moreover we may take $p' = q'$ so that multiplication is jointly continuous. In constructing the topological Hochschild homology, we use the projective tensor product $\hat{\otimes}$ instead of the usual tensor product, in order to take account of the topology of $\mathcal{A}$. This is defined by placing on the algebraic tensor product the strongest locally convex topology such that the bilinear map $(a, b) \to a\otimes b$, $\mathcal{A}\times\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ is continuous, [6, 18]. If $\mathcal{A}$ is complete with respect to this Frechet topology, then the topological tensor product is also Frechet and complete for this topology. The resulting Hochschild homology groups are still denoted by $HH_*(\mathcal{A})$. *Provided* that these groups are Hausdorff, all the important properties of Hochschild homology, including the long exact sequence, carry over to the topological setting.

The universal differential algebra over an algebra $\mathcal{A}$ is defined as follows. As an $\mathcal{A}$ bimodule, $\Omega^1(\mathcal{A})$ is generated by the symbols $\{\delta a\}_{a\in\mathcal{A}}$ subject only to the relations $\delta(ab) = a\delta(b) + \delta(a)b$. Note that this implies that $\delta(\mathbf{C}) = \{0\}$. This serves to define both left and right module structures and relates them so that, for instance,

$$a\delta(b) = \delta(ab) - \delta(a)b. \tag{25}$$

We then define

$$\Omega^n(\mathcal{A}) = \bigotimes_{i=1}^{n} \Omega^1(\mathcal{A}) \tag{26}$$

and

$$\Omega^*(\mathcal{A}) = \bigoplus_{n=0} \Omega^n(\mathcal{A}), \tag{27}$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$. Thus $\Omega^*(\mathcal{A})$ is a graded algebra, and we make it a differential algebra by setting

$$\delta(a\delta(b_1)\cdots\delta(b_k)) = \delta(a)\delta(b_1)\cdots\delta(b_k), \tag{28}$$

and

$$\delta(\omega\rho) = \delta(\omega)\rho + (-1)^{|\omega|}\omega\delta(\rho) \tag{29}$$

where ω is homogenous of degree $|\omega|$. If $\mathcal{A}$ is an involutive algebra, we make $\Omega^*(\mathcal{A})$ an involutive algebra by setting

$$(\delta(a))^* = -\delta(a^*),\ \ (\omega\rho)^* = (\rho)^*(\omega)^* \quad a \in \mathcal{A},\ \ \rho, \omega \in \Omega^*(\mathcal{A}). \tag{30}$$

With these conventions, $\Omega^*(\mathcal{A})$ is a graded differential algebra, with graded differential δ.

It turns out, [2], that the chain complex used to define Hochschild homology $HH_*(\mathcal{A})$ is the same linear space as $\Omega^*(\mathcal{A})$. So

$$C_n(\mathcal{A}) \cong \Omega^n(\mathcal{A}) \tag{31}$$

$$(a_0, a_1, ..., a_n) \to a_0\delta a_1\cdots\delta a_n. \tag{32}$$

The algebraic and differential structures of these spaces are different, however. The relation between b and δ is known, and is given by

$$b(\omega\delta a) = (-1)^{|\omega|}[\omega, a] \tag{33}$$

for $\omega \in \Omega^{|\omega|}(\mathcal{A})$ and $a \in \mathcal{A}$. We will make much use of this relation in our proof.

For a commutative algebra, there is another definition of differential forms. We retain the

definition of $\Omega^1(\mathcal{A})$, now regarded as a symmetric bimodule, but define $\widehat{\Omega}^n(\mathcal{A})$ to be $\Lambda^n_{\mathcal{A}}\Omega^1(\mathcal{A})$, the antisymmetric tensor product over $\mathcal{A}$. This has the familiar product

$$(a_0\delta a_1 \wedge \cdots \wedge \delta a_k) \wedge (b_0\delta b_1 \cdots \wedge \delta b_m) = a_0 b_0 \delta a_1 \wedge \cdots \wedge \delta a_k \wedge \delta b_1 \wedge \cdots \wedge \delta b_m \tag{34}$$

and is a differential graded algebra for δ as above. For smooth (smooth algebras are automatically unital and commutative) algebras, see [2] for this technical definition, $\widehat{\Omega}^*(\mathcal{A}) \cong HH_*(\mathcal{A})$ as graded algebras, [2], though they have different differential structures. It can also be shown that the Hochschild homology of any commutative and unital algebra $\mathcal{A}$ contains $\widehat{\Omega}^*(\mathcal{A})$ as a direct summand, [2]. For the smooth functions on a manifold, Connes' exploited the locally convex topology of the algebra $C^\infty(M)$ and the topological tensor product to prove the analogous theorem for continuous Hochschild cohomology and de Rham currents on the manifold,[6].

We also use the universal differential algebra to define connections in the algebraic setting. So suppose that E is a finite projective A module. Then it can be shown, [6], that connections, in the sense of the following definition, always exist.

Definition 1 *A (universal) connection on the finite projective A module E is a linear map $\nabla : E \to \Omega^1(A) \otimes E$ such that*

$$\nabla(a\xi) = \delta(a) \otimes \xi + a\nabla(\xi), \quad \forall a \in A,\ \xi \in E. \tag{35}$$

Note that this definition corresponds to what is usually called a universal connection, a connection being given by the same definition, but with $\Omega^1(A)$ replaced with a representation of $\Omega^1(A)$ obeying the first order condition. The distinction will not bother us, but see [3, 6]. A connection can be extended to a map $\Omega^*(A) \otimes E \to \Omega^{*+1}(A) \otimes E$ by demanding that $\nabla(\phi \otimes \xi) = \delta(\phi) \otimes \xi + (-1)^{|\phi|}(\phi \otimes 1)\nabla(\xi)$ where ϕ is homogenous of degree $|\phi|$, and extending by linearity to nonhomogenous terms.

If there is an Hermitian structure on E, and we can always suppose that there is, then we may ask what it means for a connection to be compatible with this structure. It turns out that the appropriate condition is

$$\delta(\xi, \eta)_E = (\xi, \nabla\eta)_E - (\nabla\xi, \eta)_E. \tag{36}$$

We must explain what we mean here. If we write $\nabla\xi$ as $\sum \omega_i \otimes \xi_i$, then the expression $(\nabla\xi, \eta)_E$ means $\sum(\omega_i)^*(\xi_i, \eta)_E$, and similarly for the other term. As real forms, that is differentials of self-adjoint elements of the algebra, are anti-self adjoint, we see the need for the extra minus sign in the definition. Note that some authors have the minus sign on the other term, and this corresponds to their choice of the Hermitian structure being conjugate linear in the second variable.

As a last point while on this subject, if ∇ is a connection on a finite projective $\Omega^*(A)$ module E, then $[\nabla, \cdot]$ is a connection on $\Omega^*(A)$ "with values in E". Here the commutator is the *graded* commutator, and our meaning above is made clear by

$$\begin{aligned}[\nabla, \omega] \otimes \xi &= \delta\omega \otimes \xi + (-1)^{|\omega|}\omega\nabla\xi - (-1)^{|\omega|}\omega\nabla\xi \\ &= \delta\omega \otimes \xi. \end{aligned} \tag{37}$$

This will be important later on when discussing the nature of $\mathcal{D}$.

3 Definitions and Axioms

We shall only deal with normed, involutive, unital algebras over $\mathbf{C}$ satisfying the C^*-condition: $\| aa^* \| = \| a \|^2$. To denote the algebra or ideal generated by $a_1, \ldots, a_n$, possibly subject to some relations, we shall write $\langle a_1, \ldots, a_n \rangle$. We write, for $\mathcal{H}$ a Hilbert space, $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ respectively for the bounded and compact operators on $\mathcal{H}$. We write $Cliff_{r+s}$, $Cliff_{r,s}$, for the Clifford algebra over $\mathbf{R}^{r+s}$ with Euclidean signature, respectively signature $(+\overset{r}{\cdots}+-\overset{s}{\cdots}-)$. The complex version is denoted $\mathbf{C}liff_{r+s}$. It is important to note that while our algebras $\mathcal{A}$ are imbued with a C^*-norm, we do not suppose that it is complete with respect to the topology determined by this norm, nor that this norm determines the topology of $\mathcal{A}$.

Typically in noncommutative geometry, we consider a representation of an involutive algebra

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \tag{38}$$

together with a closed unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$, chosen so that the representation of $\mathcal{A}$ extends to (bounded) representations of $\Omega^*(\mathcal{A})$. This is done by requiring

$$\pi(\delta a) = [\mathcal{D}, \pi(a)]. \tag{39}$$

So in particular, commutators of $\mathcal{D}$ with $\mathcal{A}$ must be bounded. We then set

$$\pi(\delta(a\delta(b_1)\cdots\delta(b_k))) = [\mathcal{D}, \pi(a)][\mathcal{D}, \pi(b_1)]\cdots[\mathcal{D}, \pi(b_k)]. \tag{40}$$

We say that the representation of $\Omega^*(\mathcal{A})$ is induced from the representation of $\mathcal{A}$ by $\mathcal{D}$. With the *-structure on $\Omega^*(\mathcal{A})$ described in the last section, π will be a *-morphism of $\Omega^*(\mathcal{A})$. We shall frequently write $[\mathcal{D}, \cdot] : \Omega^*(\mathcal{A}) \to \Omega^{*+1}(\mathcal{A})$, where we mean that the commutator acts only on elements of $\mathcal{A}$, not $\delta\mathcal{A}$, as defined above.

As π is only a *-morphism of $\Omega^*(\mathcal{A})$ induced by a *-morphism of $\mathcal{A}$, and not a map of differential algebras, we should not expect from the outset that it would encode the differential structure of $\Omega^*(\mathcal{A})$. In fact, it is well known that we may have the situation

$$\pi(\sum_i a^i \delta b_1^i \cdots \delta b_k^i) = 0 \tag{41}$$

while

$$\pi(\sum_i \delta a^i \delta b_1^i \cdots \delta b_k^i) \neq 0. \tag{42}$$

These nonzero forms are known as junk, [6]. To obtain a differential algebra, we must look at

$$\pi(\Omega^*(\mathcal{A}))/\pi(\delta \ker \pi). \tag{43}$$

The pejorative term junk is unfortunate, as the example of the canonical spectral triple on a spin^c manifold shows (and as we shall show later with one extra assumption on $|\mathcal{D}|$ and the representation). This is given by the algebra of smooth functions $\mathcal{A} = C^\infty(M)$ acting as multiplication operators on $\mathcal{H} = L^2(M, S)$, where S is the bundle of spinors and $\mathcal{D}$ is taken to be the Dirac operator. In this case, [6], the induced representation of the universal differential algebra is (up to a possible twisting by a complex line bundle)

$$\pi(\Omega^*(\mathcal{A})) \cong Cliff(T^*M) \otimes \mathbf{C} = \mathbf{C}liff(T^*M) \tag{44}$$

and

$$\pi(\Omega^*(\mathcal{A}))/\pi(\delta \ker \pi) \cong \Lambda^*(T^*M) \otimes \mathbf{C}. \tag{45}$$

Clearly it is the irreducible representation of the former algebra which encodes the hypothesis 'spin^c'. We will come back to this throughout the paper, and examine it more closely in Section 5. With the above discussion as some kind of motivation, let us now make some definitions.

Definition 2 *A smooth spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is given by a representation*

$$\pi : \Omega^*(\mathcal{A}) \otimes \mathcal{A}^{op} \to \mathcal{B}(\mathcal{H}) \tag{46}$$

induced from a representation of $\mathcal{A}$ *by* $\mathcal{D} : \mathcal{H} \to \mathcal{H}$ *such that*

1) $[\pi(\phi), \pi(b^{op})] = 0,\ \forall \phi \in \Omega^*(\mathcal{A}),\ b^{op} \in \mathcal{A}^{op}$

2) $[\mathcal{D}, \pi(a)] \in \mathcal{B}(\mathcal{H}),\ \forall a \in \mathcal{A}$

3) $\pi(a), [\mathcal{D}, \pi(a)] \in \cap_{m=1}^{\infty} Dom \delta^m$ *where* $\delta(x) = [|\mathcal{D}|, x]$.

Further, we require that $\mathcal{D}$ *be a closed self-adjoint operator, such that the resolvent* $(\mathcal{D}-\lambda)^{-1}$ *is compact for all* $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

The first condition here is Connes' first order condition, and it plays an important rôle in all that follows. It is usually stated as $[[\mathcal{D}, a], b^{op}] = 0$, for all $a, b \in \mathcal{A}$. A unitary change of representation on $\mathcal{H}$ given by $U : \mathcal{H} \to \mathcal{H}$ sends $\mathcal{D}$ to $U\mathcal{D}U^* = \mathcal{D} + U[\mathcal{D}, U^*]$. When it is important to distinguish between the various operators so obtained, we will write $\mathcal{D}_\pi$. Note that the smoothness condition can be encoded by demanding that the map $t \to e^{it|\mathcal{D}|} b e^{-it|\mathcal{D}|}$ is C^∞ for all $b \in \pi(\Omega^*(\mathcal{A}))$. This condition also restricts the possible form of $|\mathcal{D}|$ and so $\mathcal{D}$. This will in turn limit the possibilities for the product struture in $\Omega^*_{\mathcal{D}}(\mathcal{A})$, eventually showing us that it must be the Clifford algebra (up to a possible twisting). It is of some importance that, as mentioned, the Hochschild boundary on differential forms is given by

$$b(\omega \delta a) = (-1)^{|\omega|} [\omega, a]. \tag{47}$$

A representation of $\Omega^*(\mathcal{A})$ induced by $\mathcal{D}$ and the first order condition satisfies

$$\pi \circ b = 0. \tag{48}$$

That is, Hochschild boundaries are sent to zero by π. First this makes sense, as $HH_*(\mathcal{A})$ is a quotient of $C_*(\mathcal{A})$, which has the same linear structure as $\Omega^*(\mathcal{A})$. Second, it gives us a representation of Hochschild homology

$$\pi : HH_*(\mathcal{A}) \to \pi(\Omega^*(\mathcal{A})). \tag{49}$$

If $\pi : \mathcal{A} \to \pi(\mathcal{A})$ is faithful, then the resulting map on Hochschild homology will also be injective. The reason for this is that the kernel of π will be generated solely by elements satisfying the first order condition; i.e. Hochschild boundaries. We will discuss this matter and its ramifications further in the body of the proof.

To control the dimension we have two more assumptions.

Definition 3 *For* $p = 0, 1, 2, \ldots$, *a* (p, ∞)*-summable spectral triple is a smooth spectral triple with*

1) $|\mathcal{D}|^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$

2) a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$ *with*

$$\pi(c) = \Gamma \tag{50}$$

where if p *is odd* $\Gamma = 1$ *and if* p *is even,* $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma\pi(a) - \pi(a)\Gamma = 0$ *for all* $a \in \mathcal{A}$ *and* $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$.

Note that condition 1) is invariant under unitary change of representation. Condition 2) is a very strict restraint on potential geometries. We will write Γ or $\pi(c)$ in all dimensions unless we need to distinguish them. As a last definition for now, we define a real spectral triple.

Definition 4 *A real (p,∞)-summable spectral triple is a (p,∞)-summable spectral triple together with an anti-linear involution $J:\mathcal{H}\to\mathcal{H}$ such that*
1)$J\pi(a)^*J^* = \pi(a)^{op}$
2)$J^2=\epsilon,\ J\mathcal{D}=\epsilon'\mathcal{D}J,\ J\Gamma=\epsilon''\Gamma J$,
where $\epsilon,\epsilon',\epsilon''\in\{-1,1\}$ depend only on $p \bmod 8$ as follows:

$$\begin{array}{ccccccccc} p & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \epsilon & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ \epsilon' & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ \epsilon'' & 1 & \times & -1 & \times & 1 & \times & -1 & \times \end{array} \tag{51}$$

We will learn much about the involution from the proof, but let us say a few words. First, the map $\pi(a)\to\pi(a)^{op}$ in part 1), Definition 4, is $\mathbf{C}$ linear. Though we won't deal with the noncommutative case in any great detail in this paper, let us just point out an interesting feature. The requirement $Jb^*J^*=b^{op}$ and the first order condition allow us to say that

$$\begin{aligned}[] [\mathcal{D},\pi(a\otimes b^{op})] &= \pi(b^{op})[\mathcal{D},\pi(a)]+\pi(a)[\mathcal{D},b^{op}] \\ &= \pi(b^{op})[\mathcal{D},\pi(a)]+\epsilon'\pi(a)J[\mathcal{D},\pi(b)]J^*. \end{aligned} \tag{52}$$

This allows us to define a representation $\pi(\Omega^*(A^{op}))=\epsilon'\pi(\Omega^*(A))^{op}$. Then, just as A^{op} commutes with $\pi(\Omega^*(A))$, A commutes with $\pi(\Omega^*(A))^{op}$. In the body of the paper we introduce a slight generalisation of the operator J which allows us to introduce an indefinite metric on our manifold. We leave the details until the body of the proof. As noted in the introduction, in the commutative case we find that $\pi(\Omega^*(\mathcal{A}))$ is automatically a symmetric $\mathcal{A}$-bimodule, so when $\mathcal{A}$ is commutative, we may replace $\mathcal{A}\otimes\mathcal{A}^{op}$ by $\mathcal{A}$.

With this formulation (i.e. hiding all the technicalities in the definitions) the axioms for noncommutative geometry are easy to state. A real noncommutative geometry is a real (p,∞)-summable spectral triple satisfying the following two axioms:

i) Axiom of Finiteness and Absolute Continuity

As an $\mathcal{A}\otimes\mathcal{A}^{op}$ module, or equivalently, as an $\mathcal{A}$-bimodule, $\mathcal{H}_\infty=\cap_{m=1}^\infty\mathrm{Dom}\mathcal{D}^m$ is finitely generated and projective. Writing $\langle\cdot,\cdot\rangle$ for the inner product on $\mathcal{H}_\infty$, we require that there be given an Hermitian structure, $(\cdot,\cdot)$, on $\mathcal{H}_\infty$ such that

$$\langle a\xi,\eta\rangle = ⨍ a(\xi,\eta)|\mathcal{D}|^{-p},\ \forall a\in\pi(\mathcal{A}),\ \xi,\eta\in\mathcal{H}_\infty. \tag{53}$$

ii) Axiom of Poincaré Duality

Setting $\mu=[(\mathcal{H},\mathcal{D},\pi(c))]\in KR^p(\mathcal{A}\otimes\mathcal{A}^{op})$, we require that the cap product by μ is an isomorphism;

$$K_*(\mathcal{A})\stackrel{\cap\mu}{\to}K^*(\mathcal{A}). \tag{54}$$

Note that μ depends only on the homotopy class of π, and in particular is invariant under unitary change of representation. If we want to discuss submanifolds (i.e. unfaithful representations of $\mathcal{A}$ that satisfy the definitions/axioms) we would have to consider $\mu\in K^*(\pi(\mathcal{A}))$, and the isomorphism would be between the K-theory and the K-homology of $\pi(\mathcal{A})$. Equivalently,

we could consider $\mu \in K^*(\mathcal{A}, \ker \pi)$, the relative homology group, [9]. We will not require this degree of generality. To see that μ defines a KR class, note that $J \cdot J^*$ provides us with an involution on $\pi(\Omega^*(\mathcal{A} \otimes \mathcal{A}^{op}))$. It may or may not be trivial, but always allows us to regard the K-cycle obtained from μ as Real, in the sense of [11].

Axiom i) controls a great deal of the topological and measure theoretic structure of our geometry. Suppose $\mathcal{A}$ is a commutative algebra. Since $\overline{\pi(\mathcal{A})}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, any finite projective module over $\overline{\pi(\mathcal{A})}$ is isomorphic to a bundle of continuous sections $\Gamma(X, E)$ for some complex bundle $E \to X$. Here $X = Spec(\overline{\pi(\mathcal{A})})$. However, here we have a $\pi(\mathcal{A})$ module, and this distinction is tied up with the smoothness of the coordinates. In particular, this axiom tells us that the algebra $\mathcal{A}$ is complete with respect to the topology determined by the semi-norms $a \to \| \delta^n(a) \|$. To see this, consider the action of the completion, which we temporarily denote by $\mathcal{A}_\infty$, on $\mathcal{H}_\infty$. If $\mathcal{A}$ were not complete, then $\mathcal{A}_\infty\mathcal{H}_\infty \not\subseteq \mathcal{H}_\infty$, because being finite and projective over $\mathcal{A}$, $\mathcal{H}_\infty = e\mathcal{A}^N$, for some idempotent $e \in M_N(\mathcal{A})$ and some N. However, $\mathcal{H}_\infty$ is defined as the intersection of the domains of $\mathcal{D}^m$ for all m. In particular, $\mathcal{D}^2$ preserves $\mathcal{H}_\infty$, so that $|\mathcal{D}|$ must also. If we write $\mathcal{D} = F|\mathcal{D}| = |\mathcal{D}|F$, where F is the phase of $\mathcal{D}$, then F too must preserve $\mathcal{H}_\infty$. So let $a \in \mathcal{A}_\infty$ and $\xi \in \mathcal{H}_\infty$. Then

$$D^m a\xi = F^{m \bmod 2}|\mathcal{D}|^m a\xi \tag{55}$$

and by the boundedness of $\delta^m(a)$ for all m, we see that $D^m a\xi \in \mathcal{H}_\infty$ for all m. Hence $\mathcal{A}_\infty = \mathcal{A}$, and $\mathcal{A}$ is complete. We shall continue to use the symbol $\mathcal{A}$, and note that the completeness of $\mathcal{A}$ in the topology determined by δ makes $\mathcal{A}$ a Frechet space and allows us to use the topological version of Hochschild homology.

Axiom ii), perhaps surprisingly, is related to the Dixmier trace. Connes has shown, [6], that the Hochschild cohomology class (but importantly, *not* the cyclic class, see [15, 17]) of $ch_*([(\mathcal{H}, \mathcal{D}, \pi(c))])$ is given by ϕ_ω, where

$$\phi_\omega(a_0, a_1, \ldots, a_p) = \lambda_p \int\!\!\!\!\!- \pi(c)a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_p]|\mathcal{D}|^{-p} \tag{56}$$

for any $a_i \in \pi(\mathcal{A})$. Here $\pi(c)$ is the representation of the Hochschild cycle and λ_p is a constant. Since the K-theory pairing is non-degenerate by Poincaré Duality, $\phi_\omega \neq 0$. Thus, in particular, $\int\!\!\!\!- \pi(c)^2|\mathcal{D}|^{-p} = \int\!\!\!\!- |\mathcal{D}|^{-p} \neq 0$, and operators of the form

$$\pi(c)a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_p]|\mathcal{D}|^{-p} \tag{57}$$

are measurable; i.e. their Dixmier trace is well-defined. This also shows us , for example, that elements of the form $\pi(c)^2a|\mathcal{D}|^{-p} = a|\mathcal{D}|^{-p}$ are measurable for all $a \in \pi(\mathcal{A})$ and so defines a trace on $\pi(\mathcal{A})$. It also tells us that $|\mathcal{D}|^{-1} \notin \mathcal{L}_0^{(p,\infty)}(\mathcal{H})$, and furthermore, that the cyclic cohomology class is not in the image of the periodicity operator; i.e. p is a lower bound on the dimension of the cyclic cocycle determined by the Chern character of the Fredholm module associated to μ, [6]. For more details on K-theory and Poincaré duality, see [6, 7, 8, 9, 10].

Though the representation of $\mathcal{D}$ may vary a great deal within the K-homology class μ, the trace defined on $\mathcal{A}$ by $\int\!\!\!\!- \cdot|\mathcal{D}|^{-p}$ is invariant under unitary change of representation. Let $U \in \mathcal{B}(\mathcal{H})$ be unitary with $[\mathcal{D}, U]$ bounded. Then

$$|U\mathcal{D}U^*| = \sqrt{(U\mathcal{D}U^*)^*(U\mathcal{D}U^*)} = \sqrt{U\mathcal{D}^2U^*}. \tag{58}$$

However, $(U|\mathcal{D}|U^*)^2 = U|\mathcal{D}|^2U^* = U\mathcal{D}^2U^*$ so

$$|U\mathcal{D}U^*| = U|\mathcal{D}|U^*. \tag{59}$$

From this $(U|\mathcal{D}|U^*)^{-1} = U|\mathcal{D}|^{-1}U^*$, and

$$|U\mathcal{D}U^*|^{-p} = U|\mathcal{D}|^{-p}U^*. \tag{60}$$

It is a general property of the Dixmier trace, [6], that conjugation by bounded invertible operators does not alter the integral (this is just the trace property). So

$$⨍ |U\mathcal{D}U^*|^{-p} = ⨍ U|\mathcal{D}|^{-p}U^* = \int |\mathcal{D}|^{-p}. \tag{61}$$

Furthermore, for any $a \in \mathcal{A}$,

$$⨍ U\pi(a)U^*|U\mathcal{D}U^*|^{-p} = ⨍ \pi(a)|\mathcal{D}|^{-p}, \tag{62}$$

showing that integration is well-defined given
1) the unitary equivalence class $[\pi]$ of π
2) the choice of $c \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$.

For this reason we do not need to distinguish between $⨍ |\mathcal{D}_\pi|^{-p}$ and $⨍ |\mathcal{D}_{U\pi U^*}|^{-p}$ and we will simply write $\mathcal{D}$ in this context. Anticipating our later interest, we note that $⨍ |\mathcal{D}|^{2-p}$ is not invariant. Sending $\mathcal{D}$ to $U\mathcal{D}U^*$ sends $⨍ |\mathcal{D}|^{2-p}$ to

$$\begin{aligned} ⨍ (U\mathcal{D}U^*)^2|U\mathcal{D}U^*|^{-p} &= ⨍ (\mathcal{D} + A)^2U|\mathcal{D}|^{-p}U^* \\ &= ⨍ (\mathcal{D}^2 + \{\mathcal{D}, A\} + A^2)U|\mathcal{D}|^{-p}U^* \\ &= ⨍ (\mathcal{D}^2U|\mathcal{D}|^{-p}U^*) + ⨍ (\{\mathcal{D}, A\} + A^2)|\mathcal{D}|^{-p} \\ &= ⨍ (\mathcal{D}^2 + \{\mathcal{D}, A\} + A^2)|\mathcal{D}|^{-p} - ⨍ U[\mathcal{D}^2, U^*]|\mathcal{D}|^{-p}, \end{aligned} \tag{63}$$

where $A = U[\mathcal{D}, U^*]$ and $\{\mathcal{D}, A\} = \mathcal{D}A + A\mathcal{D}$. For this to make sense we must have $[\mathcal{D}, U^*]$ bounded of course. It is important for us that we can evaluate this using the Wodzicki residue when $\mathcal{D}$ is an operator of order 1 on a manifold. Note that when this is the case, $U[\mathcal{D}^2, U^*]$ is a first order operator, and the contribution from this term will be from the zero-th order part of a first order operator.

4 Statement and Proof of the Theorem

4.1 Statement

Theorem 9 (Connes, 1996) *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, c)$ be a real, (p,∞)-summable noncommutative geometry with $p \geq 1$ such that*

i) $\mathcal{A}$ is commutative and unital;

ii) π is irreducible (i.e. only scalars commute with $\pi(\mathcal{A})$ and $\mathcal{D}$).

Then

1) The space $X = Spec(\overline{\pi(\mathcal{A})})$ is a compact, connected, metrisable Hausdorff space for the weak topology. So $\mathcal{A}$ is separable and in fact finitely generated.*

2) Any such π defines a metric d_π on X by

$$d_\pi(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[\mathcal{D}, \pi(a)]\| \leq 1\} \tag{64}$$

and the topology defined by the metric agrees with the weak topology. Furthermore this metric depends only on the unitary equivalence class of π.*

3) The space X is a smooth spin manifold, and the metric above agrees with that defined using geodesics. For any such π there is a smooth embedding $X \hookrightarrow \mathbf{R}^N$.

4) The fibres of the map $[\pi] \to d_\pi$ are a finite collection of affine spaces A_σ parametrised by the spin structures σ on X.

5) For $p > 2$, $⨍ |\mathcal{D}_\pi|^{2-p} := WRes(|\mathcal{D}_\pi|^{2-p})$ is a positive quadratic form on each A_σ, with unique minimum π_σ.

6) The representation π_σ is given by $\mathcal{A}$ acting as multiplication operators on the Hilbert space $L^2(X, S_\sigma)$ and $\mathcal{D}_{\pi_\sigma}$ as the Dirac operator of the lift of the Levi-Civita connection to the spin bundle S_σ.

7) For $p > 2$ $⨍ |\mathcal{D}_{\pi_\sigma}|^{2-p} = -\frac{(p-2)c(p)}{12} \int_X R\sqrt{g} d^p x$ where R is the scalar curvature and

$$c(p) = \frac{2^{[p/2]}}{(4\pi)^{p/2}\Gamma(p/2+1)}. \tag{65}$$

Remark: Since, as is well known, every spin manifold gives rise to such data, the above theorem demonstrates a one-to-one correspondence (up to unitary equivalence and spin structure preserving diffeomorphisms) between spin structures on spin manifolds and real commutative geometries.

4.2 Proof of 1) and 2)

Without loss of generality, we will make the simplifying assumption that π is faithful on $\mathcal{A}$. This allows us to identify $\mathcal{A}$ with $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$, and we will simply write $\mathcal{A}$. As π is a *-homomorphism, the norm closure, $\overline{\mathcal{A}}$, is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the Gelfand-Naimark theorem tells us that $X = Spec(\overline{\mathcal{A}})$ is a compact, Hausdorff space. Since $\mathcal{A}$ is dense in its norm closure, each state on $\mathcal{A}$ (defined with respect to the C^* norm of $\mathcal{A}$) extends to a state on the closure, by continuity. Recall that we are assuming that $\mathcal{A}$ is imbued with a norm such that the C^*-condition is satisfied for elements of $\mathcal{A}$; thus here we mean continuity in the norm. Hence $Spec(\mathcal{A}) = Spec(\overline{\mathcal{A}})$. The connectivity of such a space is equivalent to the non-existence of nontrivial projections in $\overline{\mathcal{A}} \cong C(X)$. So let $p \in \overline{\mathcal{A}}$ be such that $p^2 = p$. Then

$$[\mathcal{D}, p] = [\mathcal{D}, p^2] = p[\mathcal{D}, p] + [\mathcal{D}, p]p = 2p[\mathcal{D}, p]. \tag{66}$$

So $(1-2p)[\mathcal{D},p]=0$ implying that $[\mathcal{D},p]=0$. By the irreducibility of π, we must have $p=1$ or $p=0$. Hence $\mathcal{A}$ contains no non-trivial projections and X is connected. Note that the irreducibility also implies that $[\mathcal{D},a]\neq 0$ unless a is a scalar. Also, as there are no projections, any self-adjoint element of $\mathcal{A}$ has only continuous spectrum.

The reader will easily show that equation (64) does define a metric on X, [20]. The topology defined by this metric is finer than the weak* topology, so functions continuous for the weak* topology are automatically continuous for the metric. Furthermore, elements of $\mathcal{A}$ are also Lipschitz, since for any $a\in\mathcal{A}$, $\parallel [\mathcal{D},a]\parallel\leq 1$, we have $|a(x)-a(y)|\leq d(x,y)$. Thus for any $a\in\mathcal{A}$ we have $|a(x)-a(y)|\leq\parallel[\mathcal{D},a]\parallel d(x,y)$. Later we will show that the metric and weak* topologies actually agree. This will follow from the fact that $\mathcal{A}$, and so $\overline{\mathcal{A}}$, is finitely generated. This also implies the separability of $C(X)\cong\overline{\mathcal{A}}$, which is equivalent to the metrizability of X. This will complete the proof of 1) and 2), but it will have to wait until we have learned some more about $\mathcal{A}$. The last point of 2) is that the metric is invariant under unitary transformations. That is if $U:\mathcal{H}\to\mathcal{H}$ is unitary

$$[U\mathcal{D}U^*,UaU^*]=U[\mathcal{D},a]U^*\Rightarrow\parallel[U\mathcal{D}U^*,UaU^*]\parallel=\parallel[\mathcal{D},a]\parallel. \tag{67}$$

So while $\mathcal{D}$ will be changed by a unitary change of representation

$$\mathcal{D}:=\mathcal{D}_\pi\to U\mathcal{D}U^*:=\mathcal{D}_{U\pi U^*}=\mathcal{D}_\pi+U[\mathcal{D}_\pi,U^*], \tag{68}$$

commutators with $\mathcal{D}$ change simply. For this reason, when we only need the unitary equivalence class of π, we drop the π, and write $\Omega^*_{\mathcal{D}}(\mathcal{A})$ for $\pi(\Omega^*(\mathcal{A}))$, where the $\mathcal{D}$ is there to remind us that this is the representation of $\Omega^*(\mathcal{A})$ induced by $\mathcal{D}$ and the first order condition.

4.3 Proof of 3) and remainder of 1) and 2)

Before beginning the proof of 3), which will also complete the proof of 1) and 2), let us outline our approach, as this is the longest, and most important, portion of the proof.

4.3.1 **Generalities**. This section deals with the various bundles involved, their Hermitian structures and their relationships. We also analyse the structure of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ and $\Lambda^*_{\mathcal{D}}(\mathcal{A})$, particularly in relation to Hochschild homology.

4.3.2 X **is a p-dimensional topological manifold**. We show that the elements of $\mathcal{A}$ involved in the Hochschild cycle

$$\pi(c)=\Gamma=\sum_i a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i] \tag{69}$$

provided by the axioms generate $\mathcal{A}$. This is done in two steps. Results from 4.3.1 show that Γ is antisymmetric in the $[\mathcal{D},a_j^i]$, and this is used to show that $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is finitely generated by the $[\mathcal{D},a_j^i]$ appearing in Γ. The second step then uses the long exact sequence in Hochschild homology to show that $\mathcal{A}$ is finitely generated by the a_j^i and $1\in\mathcal{A}$. In the process we will also show that X is a topological manifold.

4.3.3 X **is a smooth manifold**. We show here that $\mathcal{A}$ is $C^\infty(X)$, and in particular that X is a smooth manifold. After proving that the weak* and metric topologies on X agree, we show that $\mathcal{A}$ is closed under the holomorphic functional calculus, so that the K-theories of $\mathcal{A}$ and $C(X)$ agree. At this point we will have completed the proof of 1) and 2).

4.3.4 X **is a spin^c manifold**. The form of the operators $\mathcal{D}$, $|\mathcal{D}|$ and $\mathcal{D}^2$ is investigated. The main result is that $\mathcal{D}^2$ is a generalised Laplacian while $\mathcal{D}$ is a generalised Dirac operator, in the sense of [27]. This allows us to show that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is (at least locally) the Clifford algebra of

the complexified cotangent bundle. This is sufficient to show that the metric given by equation (64) agrees with the geodesic distance on X. As the representation of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is irreducible, we will have completed the proof that X is a spin^c manifold.

4.3.5 X **is spin**. It is at this point that we utilise the real structure. Furthermore, we reformulate Connes' result to allow a representation of the Clifford algebra of an indefinite metric. This will necessarily involve a change in the underlying topology, which we do not investigate here.

4.3.1 Generalities.

The axiom of finiteness and absolute continuity tells us that $\mathcal{H}_\infty = \cap_{m\geq 1} Dom\mathcal{D}^m$ is a finite projective $\mathcal{A}$ module. This tells us that $\mathcal{H}_\infty \cong e\mathcal{A}^N$, as an $\mathcal{A}$ module, for some N and some $e = e^2 \in M_N(\mathcal{A})$. Furthermore, from what we know about $\mathcal{A}$ and $Spec(\mathcal{A})$, $\mathcal{H}_\infty$ is also isomorphic to a bundle of sections of a vector bundle over X, say $\mathcal{H}_\infty \cong \Gamma(X, S)$. These sections will be of some degree of regularity which is at least continuous as $\mathcal{A} \subset C(X)$. This bundle is also imbued with an Hermitian structure $(\cdot,\cdot)_E : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{A}$ such that $(a\psi, b\eta)_S = a^*(\psi,\eta)_S b$ etc, which provides us with an interpretation of the Hilbert space as $\mathcal{H} = L^2(X, S, ⨍(\cdot,\cdot)|\mathcal{D}|^{-p})$. We will return to the important consequences of the Hermitian structure and the measure theoretic niceties of the above interpretation later.

As mentioned earlier, $\mathcal{A}$ is a Frechet space for the locally convex topology coming from the family of seminorms

$$\| a \|, \quad \| \delta(a) \|, \quad \| \delta^2(a) \|, \ldots \quad \forall a \in \mathcal{A}, \quad \delta(a) = [|\mathcal{D}|, a]. \tag{70}$$

Note that the first semi-norm in this family is the C^*-norm of $\mathcal{A}$, and that $\delta^n(a)$ makes sense for all $a \in \mathcal{A}$ by hypothesis. As the first semi-norm is in fact a norm, this topology is Hausdorff. In fact our hypotheses allow us to extend these seminorms to all of $\Omega^*_{\mathcal{D}}(\mathcal{A})$, and it too will be complete for this topology.

Now let us turn to the differential structure. The first things to note are that $\mathcal{D}\mathcal{H}_\infty \subset \mathcal{H}_\infty$, $\mathcal{A}\mathcal{H}_\infty \subset \mathcal{H}_\infty$ and $\| [\mathcal{D}, a] \| < \infty \ \forall a \in \mathcal{A}$. The associative algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$, so $\Omega^*_{\mathcal{D}}(\mathcal{A})\mathcal{H}_\infty \subset \mathcal{H}_\infty$. In other words

$$\Omega^*_{\mathcal{D}}(\mathcal{A}) \subset End(\Gamma(X,S)) \cong End(e\mathcal{A}^N) \cong \{B \in M_N(\mathcal{A}) : Be = eB\}. \tag{71}$$

The most important conclusion of these observations is that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ and so $\Omega^1_{\mathcal{D}}(\mathcal{A})$ are finite projective over $\mathcal{A}$, and so are both (sections of) vector bundles over X, the former being (the sections of) a bundle of algebras as well. Moreover, the irreducibility hypothesis tells us that $\Gamma(X, S)$ is an irreducible module (of sections) for the algebra (of sections) $\Omega^*_{\mathcal{D}}(\mathcal{A})$.

The finite generation of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ can be used to show that the algebra $\mathcal{A}$ is finitely generated. We will then be dealing with a finitely generated algebra and this can be used to show that the weak* and metric topologies agree. When we have completed the proofs of these matters, 1) will have been proved completely.

So what is $\Omega^*_{\mathcal{D}}(\mathcal{A})$? The central idea for studying this algebra is the first order condition. When we construct this representation of $\Omega^*(\mathcal{A})$ from π and $\mathcal{A}$ using $\mathcal{D}$, the first order condition forces us to identify the left and right actions of $\mathcal{A}$ on $\Omega^*_{\mathcal{D}}(\mathcal{A})$, at least in the commutative case. Assuming as we are that the representation is faithful on $\mathcal{A}$, we see that the ideal $\ker \pi$ is generated by the first order condition,

$$\ker \pi = \langle \omega a - a\omega \rangle_{a\in\mathcal{A}, \omega\in\Omega^*(\mathcal{A})} = \langle \text{first order condition} \rangle. \tag{72}$$

So for $\omega = \delta f$ of degree 1 and $a \in \mathcal{A}$, $a\delta f - \delta f a \in \ker\pi$ and

$$(\delta f)(\delta a) + (\delta a)(\delta f) \in \delta \ker\pi. \tag{73}$$

Equation (72) ensures that $\pi \circ b = 0$, as $\mathrm{Image}(b) = \ker\pi$, so that we have a well-defined faithful representation of Hochschild homology

$$\pi : HH_*(\mathcal{A}) \to \Omega^*_{\mathcal{D}}(\mathcal{A}). \tag{74}$$

If we write $d = [\mathcal{D}, \cdot]$, we see that the existence of junk is due to the fact that $\pi \circ \delta \neq d \circ \pi$, and that this may be traced directly to the first order condition. Let us continue to write $\Omega^*_{\mathcal{D}}(\mathcal{A}) := \pi(\Omega^*(\mathcal{A}))$ and also write $\Lambda^*_{\mathcal{D}}(\mathcal{A}) := \pi(\Omega^*(\mathcal{A}))/\pi(\delta\ker\pi)$, and note that the second algebra is skew-commutative, and a graded differential algebra for the differential $d = [\mathcal{D}, \cdot]$. Note that this notation differs somewhat from the usual, [6]. Note that $\Omega^1_{\mathcal{D}}(\mathcal{A})$ and $\Lambda^1_{\mathcal{D}}(\mathcal{A})$ are the same finite projective $\mathcal{A}$ module, and we denote them both by $\Gamma(X, E)$ for some bundle $E \to X$, where as before we do not specify the regularity of the sections, only that they are at least continuous for the weak* topology and Lipschitz for the metric topology.

The next point to examine is $\delta\ker\pi = \mathrm{Image}(\delta \circ b)$. This is easily seen to be generated by $\ker\pi = \mathrm{Image}(b)$ and graded commutators $(\delta a)\omega - (-1)^{|\omega|}\omega\delta a$, for $\omega \in \Omega^*(\mathcal{A})$ and $a \in \mathcal{A}$. Thus the image of $\pi \circ \delta \circ b$ in $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is junk, and this is graded commutators. As elements of the form appearing in equation (73) generate $\pi(\delta\ker\pi)$, it is useful to think of equation (73) as a kind of 'pre-Clifford' relation. In particular, controlling the representation of elements of $\delta\ker\pi$ will give rise to a representation of the Clifford algebra as well as the components of the metric tensor. More on that later.

To help our analysis, define $\sigma : \Omega^*(\mathcal{A}) \to \Omega^*(\mathcal{A})$ by $\sigma(a) = a$ for all $a \in \mathcal{A}$ and $\sigma(\omega\delta a) = (-1)^{|\omega|}(\delta a)\omega$, for $|\omega| \geq 0$. Then, [2], we have $b \circ \delta + \delta \circ b = 1 - \sigma$, on $\Omega^*(\mathcal{A})$. As $\pi \circ b = 0$, and $\mathrm{Image}(\pi \circ \delta \circ b)$ =Junk, we have $\mathrm{Image}(\pi \circ (1 - \sigma))$ =Junk.

Thus while

$$\pi(\Omega^*(\mathcal{A})) = \Omega^*_{\mathcal{D}}(\mathcal{A}) \cong \Omega^*(\mathcal{A})/\langle \mathrm{Image}(b)\rangle, \tag{75}$$

passing to the junk-free situation gives

$$\Omega^*(\mathcal{A})/\langle \mathrm{Image}(b), \mathrm{Image}(1 - \sigma)\rangle \cong \Lambda^*_{\mathcal{D}}(\mathcal{A}) \cong \widehat{\Omega}^*(\mathcal{A}). \tag{76}$$

It is easy to see that $b(1 - \sigma) = (1 - \sigma)b$, so that $\ker b$ is preserved by $1 - \sigma$. In fact, $1 - \sigma$ sends Hochschild cycles to Hochschild boundaries. For if $bc = 0$ for some element $c \in C_n(\mathcal{A})$, then

$$(1 - \sigma)c = (b\delta + \delta b)c = b\delta c \tag{77}$$

which is a boundary. So $\ker b$ is mapped into $\mathrm{Image}(b)$ under $1 - \sigma$ and so when we quotient by $\mathrm{Image}(1 - \sigma)$ we do not lose any Hochschild cycles.

So, π descends to a faithful representation of Hochschild homology with values in $\Lambda^*_{\mathcal{D}}(\mathcal{A})$. In general, the Hochschild homology groups of a commutative and unital algebra contain $\widehat{\Omega}^*(\mathcal{A})$ as a direct summand, [2], but we have shown that in fact

$$HH_*(\mathcal{A}) \cong \Lambda^*_{\mathcal{D}}(\mathcal{A}) \cong \widehat{\Omega^*}(\mathcal{A}). \tag{78}$$

This is certainly a necessary condition for the algebra $\mathcal{A}$ to be smooth, but more important for us at this point is that all Hochschild cycles are antisymmetric in elements of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. In particular, $\pi(c) = \Gamma \neq 0$ in $\Lambda^p_{\mathcal{D}}(\mathcal{A})$ and is totally antisymmetric.

There is another consequence of this result. If we define an Hermitian form on $\Omega^1_{\mathcal{D}}(\mathcal{A})$ by $(da, db)_{\Omega^1} = \frac{1}{M} Trace((da)^* db)$, then

$$(da\xi, db\eta)_S = (da, db)_{\Omega^1}(\xi, \eta)_S, \tag{79}$$

where M is the fibre dimension of the bundle underlying $\Omega^1_{\mathcal{D}}(\mathcal{A})$, and this uses the antisymmetry in an essential way. It also shows that the trace is a nondegenerate Hermitian form on $\Omega^1_{\mathcal{D}}(\mathcal{A})$.

4.3.2 X is a p-dimensional topological manifold.

We claim that the elements a^i_j, $i = 1, ..., n$ $j = 1, ..., p$ appearing in the Hochschild cycle Γ, along with $1 \in \mathcal{A}$, generate $\mathcal{A}$ as an algebra over $\mathbf{C}$. Without loss of generality we take a^i_j to be self-adjoint for $i, j \geq 1$. Furthermore, we may also assume that $\| [\mathcal{D}, a^i_j] \| = 1$. To show that the a^i_j generate, we first show that the $[\mathcal{D}, a^i_j]$ generate $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Let Γ be the (totally antisymmetric) representative of the Hochschild p-cycle provided by the axioms. We write $da := [\mathcal{D}, a]$ for brevity, and similarly we write d for the action of $[\mathcal{D}, \cdot]$ on forms.

Recall that $\Gamma = \pi(c)$, and note that for any $a \in \mathcal{A}$

$$\begin{aligned}\pi(1-\sigma)(c\delta a) &= \Gamma da - (-1)^p da\Gamma \\ &= (1 - (-1)^p(-1)^{p-1})\Gamma da \\ &= 2\Gamma da.\end{aligned} \tag{80}$$

Thus Γda is junk and so contains a symmetric factor, and $da \wedge \Gamma = 0$ for all $a \in \mathcal{A}$. In order to show that the da^i_j generate $\Omega^1_{\mathcal{D}}(\mathcal{A})$ as an $\mathcal{A}$ bimodule, we need to show that $da \wedge \Gamma = 0$ implies that da is a linear combination of the da^i_j. To do this, first write

$$\Gamma = \sum_{i=1}^n a^i_0 da^i_1 \cdots da^i_p = \sum_{i=1}^n \Gamma^i. \tag{81}$$

Now suppose that $n = 1$, so that $\Gamma = a_0 da_1 \cdots da_p$. Then if

$$(da \wedge \Gamma)(x) = (a_0 da \wedge da_1 \cdots da_p)(x) = 0 \tag{82}$$

for all $x \in X$, elementary exterior algebra tells us that $da(x)$ is a linear combination of $da_1(x), ..., da_p(x)$ in each fibre.

So if $n > 1$, the only thing we need to worry about is cancellation in the sum

$$\sum da \wedge \Gamma^i. \tag{83}$$

Without loss of generality, we can assume that at each $x \in X$ there is no cancellation in the sum

$$\sum \Gamma^i. \tag{84}$$

So for all $I, J \subset \{1, ..., n\}$ with $I \cap J = \emptyset$,

$$\sum_{i \in I} \Gamma^i(x) \neq -\sum_{j \in J} \Gamma^j(x). \tag{85}$$

If there were such terms we could simply remove them anyway, and we know in doing so we do not remove all the terms Γ^i as $\Gamma(x) \neq 0$ for all $x \in X$.

Now suppose that for some $x \in X$ and some $I, J \subset \{1, ..., n\}$ with $I \cap J = \emptyset$ we have

$$(\sum_{i \in I} da \wedge \Gamma^i)(x) = -(\sum_{j \in J} da \wedge \Gamma^j)(x). \tag{86}$$

If $da(x)$ is a linear combination of any of the terms appearing in these Γ^i's, we are done. So supposing that da is linearly independent of the terms appearing in $\sum_{I \cup J} \Gamma^i$, we have

$$\sum_{i \in I} \Gamma^i(x) = -\sum_{j \in J} \Gamma^j(x) \tag{87}$$

contradicting our assumption on Γ. Thus we may assume that no terms cancel, which shows that

$$(da \wedge \Gamma^i)(x) = 0 \tag{88}$$

for each $i = 1, ..., n$. Hence if $\Gamma^i(x) \neq 0$, $da(x)$ is linearly dependent on $da_1^i(x), ..., da_p^i(x)$. This also shows that if Γ^i, Γ^j are both nonzero at $x \in X$, then they are linearly dependent at x. These considerations show that the da_j^i generate $\Omega_{\mathcal{D}}^1(\mathcal{A})$ as an $\mathcal{A}$ bimodule.

As a consequence, the da_j^i also generate $\Lambda_{\mathcal{D}}^*(\mathcal{A})$ as a graded differential algebra. From what we have already shown, this algebra is precisely

$$\Lambda_{\mathcal{A}}^* \Omega_{\mathcal{D}}^1(\mathcal{A}) = \Lambda_{\mathcal{A}}^* \Gamma(E) = \Gamma(\Lambda^* E). \tag{89}$$

Now the da_j^i generate and any $p+1$ form in them is zero from the above argument, while we know that $\Lambda_{\mathcal{D}}^p(\mathcal{A}) \neq \{0\}$ because $\Gamma \in \Lambda_{\mathcal{D}}^p(\mathcal{A})$. Also, for all $x \in X$, we know that $\Gamma(x) \neq 0$, so each fibre $\Lambda^p E_x$ is nontrivial. Lastly, using the antisymmetry and non-vanishing of Γ, it is easy to see that for all $x \in X$ there is an i such that the $da_j^i(x)$, $j = 1, ..., p$, are linearly independent in E_x. For if , say,

$$da_1^i(x) = \sum_{j=2}^{p} c_j da_j^i(x) \tag{90}$$

then inserting this expression into the formula for Γ and using the antisymmetry shows that

$$da_1^i da_2^i \cdots da_p^i(x) = 0. \tag{91}$$

If this happenned for all i at some $x \in X$ we would have a contradiction of the non-vanishing of $\Gamma(x)$. Hence we can always find such an i.

Putting all these facts together, and recalling that X is connected, we see that E has rank p as a vector bundle, and moreover, for all $x \in X$ there is an index i such that the $da_j^i(x)$ form a basis of E_x. Later we will see that E is essentially the (complexified) cotangent bundle.

We now have the pieces necessary to show that $\mathcal{A}$ is in fact finitely generated by the a_j^i. Suppose that the functions a_j^i do not separate the points of X. Define an equivalence relation on X by

$$x \sim y \Leftrightarrow a_j^i(x) = a_j^i(y) \quad \forall i, j. \tag{92}$$

Then by adding constants to the a_j^i if necessary, there is an equivalence class B such that

$$a_j^i(B) = \{0\} \quad \forall i, j. \tag{93}$$

So the a_j^i generate an ideal $\langle a_j^i \rangle$ whose norm closure is $C_0(X \setminus B)$. The fact that $\Lambda_{\mathcal{D}}^*(\mathcal{A})$ is complete in the topology determined by the family of seminorms provided by δ, and is a locally

convex Hausdorff space for this topology, shows that the topological Hochschild homology is Hausdorff, and allows us to use the long exact sequence in topological Hochschild homology. We have the exact sequence

$$0 \to \langle a^i_j \rangle \xrightarrow{I} \mathcal{A} \xrightarrow{P} \mathcal{A}/\langle a^i_j \rangle \to 0 \tag{94}$$

as well as a norm closed version

$$0 \to C_0(X \setminus B) \to C(X) \to C(B) \to 0 \tag{95}$$

The former sequence, being a sequence of locally convex algebras, induces a long exact sequence in topological Hochschild homology. The bottom end of this looks like

$$\cdots \to \Lambda^2(\mathcal{A}/\langle a^i_j \rangle) \to \Lambda^1_{\mathcal{D}}(\langle a^i_j \rangle) \to \Lambda^1_{\mathcal{D}}(\mathcal{A}) \to \Lambda^1_{\mathcal{D}}(\mathcal{A}/\langle a^i_j \rangle) \to \langle a^i_j \rangle \xrightarrow{I} \mathcal{A} \xrightarrow{P} \mathcal{A}/\langle a^i_j \rangle \to 0. \tag{96}$$

From what we have shown, every element of $\Lambda^n_{\mathcal{D}}(\mathcal{A})$ is of the form

$$\omega = \sum (a \hat{\otimes} a_1 \hat{\otimes} a_2 \hat{\otimes} \cdots \hat{\otimes} a_n) \tag{97}$$

where $a \in \mathcal{A}$ and $a_k \in \langle a^i_j \rangle$ for each $1 \leq k \leq n$. The map induced on homology by $P : \mathcal{A} \to \mathcal{A}/\langle a^i_j \rangle$ is easy to compute:

$$P_* \sum (a \hat{\otimes} a_1 \hat{\otimes} \cdots \hat{\otimes} a_n) = \sum (P(a) \hat{\otimes} P(a_1) \hat{\otimes} \cdots \hat{\otimes} P(a_n)) = \sum (P(a) \hat{\otimes} 0 \cdots \hat{\otimes} 0) = 0. \tag{98}$$

So $\Lambda^n_{\mathcal{D}}(\mathcal{A}/\langle a^i_j \rangle) = 0$ for all $n \geq 1$. The case $n = 1$ says that

$$\delta P(a) = 1 \otimes P(a) - P(a) \otimes 1 = 0 \Rightarrow P(a) \in \mathbf{C} \cdot 1. \tag{99}$$

Hence $C(B) = \mathbf{C}$ and $B = \{pt\}$. As an immediate corollary we see that all the equivalence classes of $\sim$ are singletons, so $\mathcal{A}$ is generated in its Frechet topology by the elements a^i_j.
Now take the natural open cover of X given by the open sets

$$U^i = \{x \in X : [\mathcal{D}, a^i_1], ..., [\mathcal{D}, a^i_p] \neq 0\}. \tag{100}$$

From what we have already shown, over this open set we obtain a local trivialisation

$$E|_{U^i} \cong U^i \times \mathbf{C}^p. \tag{101}$$

As

$$|a^i_j(x) - a^i_j(y)| \leq \| [D, a^i_j]|_F \| d(x, y) \tag{102}$$

where F is any closed set containing x and y, we see that the a^i_j are constant off U^i. By altering these functions by adding scalars, we see that we can take their value off U^i to be zero. Thus $\langle a^i_j \rangle_j \subseteq C_0(U^i)$. Noting that the da^i_j provide a generating set for $\Omega^1_{\mathcal{D}}(\mathcal{A}_{U^i})$ over $\mathcal{A}_{U^i}$ (the closure of the functions in $\mathcal{A}$ vanishing off U^i for the Frechet topology), the previous argument shows that the a^i_j generate $\mathcal{A}_{U^i}$ in the Frechet topology and $C_0(U^i)$ in norm. The inessential detail that $\mathcal{A}_{U^i}$ is not unital may be repaired by taking the one point compactification of U^i or simply noting that the above argument runs as before, but now the only scalars are zero, whence the equivalence class B is empty.
We are now free to take as coordinate charts (U^i, a^i) where $a^i = (a^i_1, ..., a^i_p) : U^i \to \mathbf{R}^p$. As both the a^i_j and the a^k_j generate the functions on $U^i \cap U^k$, we may deduce the existence of continuous transition functions $f^i_{jk} : \mathbf{R}^p \to \mathbf{R}^p$ with compact support such that

$$a^i_j = f^i_{jk}(a^k_1, ..., a^k_p) \text{ on the set } U^i \cap U^k. \tag{103}$$

As these functions are necessarily continuous, we have shown that X is a topological manifold, and moreover the map $a = (a^1, ..., a^n) : X \to \mathbf{R}^{np}$ is a continuous embedding.

4.3.3 X is a smooth manifold

We can now show that X is a smooth manifold. On the intersection $U^i \cap U^k$, the functions can be taken to be generated by either $a_1^i, ..., a_p^i$ or $a_1^k, ..., a_p^k$. Thus we may write the transition functions as

$$a_j^i = f_{jk}^i = \sum_{N=0}^{\infty} p_N(a_j^k) \tag{104}$$

where the p_N are homogenous polynomials of total degree N in the a_j^k. As the a_j^i generate $\mathcal{A}$ in its Frechet topology, we may assume that this sum is convergent for all the seminorms $\| \delta^n(\cdot) \|$. Also, $\Omega_{\mathcal{D}}^*(\mathcal{A}) \subset \cap_{n\geq 1} Dom\delta^n$ and $[\mathcal{D}, \cdot] : \Omega_{\mathcal{D}}^*(\mathcal{A}) \to \Omega_{\mathcal{D}}^*(\mathcal{A})$, showing that the sequence

$$\sum_{N=0}^{\infty} [\mathcal{D}, p_N] \tag{105}$$

converges. Since $\mathcal{D}$ is a closed operator, the derivation $[\mathcal{D}, \cdot]$ can be seen to be closed as well. Thus, over the open set $U^i \cap U^k$, we see that the above sequence converges to $[\mathcal{D}, a_j^i]$, so

$$[\mathcal{D}, a_j^i] = \sum_{l=1}^{p} \sum_{N=0}^{\infty} \frac{\partial p_N}{\partial a_l^k} [\mathcal{D}, a_l^k], \tag{106}$$

where we have also used the first order condition. Consequently, the functions

$$\sum_{N=0}^{\infty} \frac{\partial p_N}{\partial a_l^k} \in \mathcal{A} \subset C(X) \tag{107}$$

are necessarily continuous. This allows us to identify

$$\frac{\partial f_{jk}^i}{\partial a_l^k} = \sum_{N=0}^{\infty} \frac{\partial p_N}{\partial a_l^k}. \tag{108}$$

Applying the above argument repeatedly to the functions $\frac{\partial f_{jk}^i}{\partial a_l^k}$, shows that f_{jk}^i is a C^∞ function. Hence X is a smooth manifold for the metric topology and $\mathcal{A} \subseteq C^\infty(X)$. In particular, the functions a_j^i are smooth.

Conversely, let $f \in C^\infty(X)$. Over any open set $V \subset U^i$ we may write

$$f = \sum_{n=0}^{\infty} p_N(a_1, .., a_p) \tag{109}$$

where we have temporarily written $a_j := a_j^i$. As f is smooth, all the sequences

$$\sum_{|\alpha|=n} \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} a_1 \cdots \partial^{\alpha_p} a_p} = \sum_{|\alpha|=n} \sum_{N=0}^{\infty} \frac{\partial^{|\alpha|} p_N}{\partial^{\alpha_1} a_1 \cdots \partial^{\alpha_p} a_p} \tag{110}$$

converge, where $\alpha \in \mathbf{N}^n$ is a multi-index. Let $p_N = \sum_{|\alpha|=N} C_\alpha a_1^{\alpha_1} \cdots a_p^{\alpha_p}$ and let $s_M = \sum_{N=0}^{M} p_N$ be the partial sum. Then

$$[|\mathcal{D}|, s_M] = \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_N a_1^{n_1} \cdots a_j^{n_j-k} [|\mathcal{D}|, a_j] a_j^{k-1} \cdots a_p^{n_p}$$

$$= \sum_{n=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j - 1} \cdots a_p^{\alpha_p} [|\mathcal{D}|, a_j]$$
$$+ \sum_{n=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j - k} [[|\mathcal{D}|, a_j], a_j^{k-1} \cdots a_p^{\alpha_p}]$$
$$= G_M^1 + \sum_{j=1}^{p} \frac{\partial s_M}{\partial a_j} [|\mathcal{D}|, a_j]. \tag{111}$$

To show that $f \in Dom\delta$, we must show that G_M^1 can be bounded independent of M, the other term being convergent by the smoothness of f and the boundedness of $[|\mathcal{D}|, a_j]$ for each j. We have the following bound

$$\| G_M^1 \| \leq \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} \| C_\alpha a_1^{\alpha_1} \cdots a_j^{\alpha_j - k} [[|\mathcal{D}|, a_j], a_j^{k-1} \cdots a_p^{\alpha_p}] \|$$
$$\leq \sum_{N=0}^{M} \sum_{|\alpha|=N} \sum_{j=1}^{p} \sum_{k=1}^{n_j} 2|C_N| \| a_1 \|^{n_1} \cdots \| a_j \|^{n_j - 1} \cdots \| a_p \|^{n_p} \| [|\mathcal{D}|, a_j] \|$$
$$= 2 \sum_{N=0}^{M} \sum_{j=1}^{p} \frac{\partial \tilde{p}_N}{\partial a_j} (\| a_1 \|, ..., \| a_p \|) \| [|\mathcal{D}|, a_j] \|, \tag{112}$$

where $\tilde{p}_N = \sum_{|\alpha|=N} |C_\alpha| a_1^{\alpha_1} \cdots a_p^{\alpha_p}$. The absolute convergence of the sequence of real numbers

$$\sum_{N=0}^{\infty} \frac{\partial p_N}{\partial a_j} (\| a_1 \|, ..., \| a_p \|) \tag{113}$$

now shows that $\| G_M^1 \|$ can be bounded independtly of M. Thus the sequence $[|\mathcal{D}|, s_M]$ converges, and as $[|\mathcal{D}|, \cdot]$ is a closed derivation, it converges to $[|\mathcal{D}|, f]$. Hence $\| [|\mathcal{D}|, f] \| < \infty$, and $f \in Dom\delta$.

Applying δ twice gives

$$[|\mathcal{D}|, [|\mathcal{D}|, s_M]] = G_M^2 + \sum_{|\alpha|=2} \frac{\partial^2 s_M}{\partial^{\alpha_j} a_j \partial^{\alpha_k} a_k} [|\mathcal{D}|, a_j]^{\alpha_j} [|\mathcal{D}|, a_k]^{\alpha_k} + \sum_{|\alpha|=1} \frac{\partial s_M}{\partial a_j} \delta^2(a_j). \tag{114}$$

The second two terms can be bounded independently of M by the smoothness of f. The term G_M^2 is a sum of commutators and double commutators which can be bounded independently of M in exactly the same manner as G_M^1. This shows that

$$\| [|\mathcal{D}|, [|\mathcal{D}|, f]] \| < \infty \tag{115}$$

and $f \in Dom\delta^2$. Continuing this line of argument shows that $f \in Dom\delta^n$ for all n, and so $f \in \mathcal{A}$. Consequently, $\mathcal{A} = C^\infty(X)$, and the seminorms $\| \delta^n(\cdot) \|$ determine the C^∞ topology on $\mathcal{A}$.

To show that the weak* and metric topologies agree, it is sufficient to show that convergence in the weak* topology implies convergence in the metric topology, as the metric topology is automatically finer.

So let $\{\phi_k\}_{k=1}^{\infty}$ be a weak* convergent sequence of pure states of $\mathcal{A}$ (or $\overline{\mathcal{A}}$). Thus there is a pure state ϕ such that for all $f \in \mathcal{A}$,

$$|\phi_k(f) - \phi(f)| \to 0. \tag{116}$$

As $\mathcal{A}$ is commutative, we know that every pure state is a $*$-homomorphism, and writing the generating set of $\mathcal{A}$ as $a_1, ..., a_{np}$ we have for $f = \sum p_N$,

$$\phi_k(f) = \sum_{N=0}^{\infty} p_N(\phi_k(a_i)) \tag{117}$$

and this makes sense since the sum is convergent in norm.

The next aspect to address is the norm of $[\mathcal{D}, f]$. Recalling that $\| [\mathcal{D}, a_i] \| = 1$, we have

$$\begin{aligned}
\| [\mathcal{D}, f] \|^2 &= \| \sum_{i,j=1}^{np} \frac{\partial f}{\partial a_i}[\mathcal{D}, a_i](\frac{\partial f}{\partial a_j})^*[\mathcal{D}, a_j]^* \| \\
&\leq \sup_{x\in X} | \sum_{i,j=1}^{np} \frac{\partial f}{\partial a_i}(x)(\frac{\partial f}{\partial a_j})^*(x)| \\
&= \sup_{a(x)\in a(X)} | \sum_{i,j=1}^{np} \frac{\partial f}{\partial x_i}(a(x))(\frac{\partial f}{\partial x_j})^*(a(x))| \\
&= \| df \|^2, \text{ regarding } f : \mathbf{R}^{np} \to \mathbf{C},
\end{aligned} \tag{118}$$

where $a : X \to \mathbf{R}^{np}$ is our (smooth) embedding and x_i are coordinates on $\mathbf{R}^{np}$. Thus $\| df \| \leq 1 \Rightarrow \| [\mathcal{D}, f] \| \leq 1$. Any function $f : \mathbf{R}^{np} \to \mathbf{C}$ satisfying $\| df \| \leq 1$ is automatically Lipschitz (as a function on $\mathbf{R}^{np}$). So

$$\begin{aligned}
|\phi_k(f) - \phi(f)| &= |f(\phi_k(a_i)) - f(\phi(a_i))| \\
&\leq |\phi_k(a_i) - \phi(a_i)| \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{119}$$

Hence

$$\begin{aligned}
\sup\{|\phi_k(f) - \phi(f)| :\| [\mathcal{D}, f] \| \leq 1\} &= \sup\{|f(\phi_k(a_i)) - f(\phi(a_i))| :\| df \| \leq 1\} \\
&\leq \{|\phi_k(a_i) - \phi(a_i)|\} \to 0
\end{aligned} \tag{120}$$

so $\phi_k \to \phi$ in the metric. So the two topologies agree.

As a last note on these issues, it is important to point out that $\mathcal{A}$ is stable under the holomorphic functional calculus. If $f : X \to \mathbf{C}$ is in $\mathcal{A}$, then we may (locally) regard it as a smooth function $f : \mathbf{R}^p \to \mathbf{C}$ of $a_1^i, ..., a_p^i$ for some i. So let $g : \mathbf{C} \to \mathbf{C}$ be holomorphic. Then

$$g \circ f \circ a^i \tag{121}$$

is patently a smooth function on X. Thus the K-theory and K-homology of $\mathcal{A}$ and $\overline{\mathcal{A}}$ coincide, [6].

4.3.4 X is a spin^c manifold

We have been given an Hermitian structure on $\mathcal{H}_\infty$, $(\cdot,\cdot)_S$, and as $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is finite projective, we are free to choose one for it also. Regarding $\Omega^*_{\mathcal{D}}(\mathcal{A})$ as a subalgebra of $End(\mathcal{H}_\infty)$, any non-degenerate Hermitian form we choose is unitarily equivalent to $([\mathcal{D},a],[\mathcal{D},b])_{\Omega^1} := \frac{1}{p}Tr([\mathcal{D},a]^*[\mathcal{D},b])$, where p is the fibre dimension of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. We have shown this is a non-degenerate positive definite quadratic form. Over each U^i, we have a local trivialisation (recalling that we have set $\Omega^1_{\mathcal{D}}(\mathcal{A}) = \Gamma(X,E)$)

$$E|_{U^i} \cong U^i \times \mathbf{C}^p. \tag{122}$$

As X is a smooth manifold, we can also define the cotangent bundle, and as the a^i are local coordinates on each U^i, we have

$$T^*_{\mathbf{C}}X|_{U^i} \cong U^i \times \mathbf{C}^p. \tag{123}$$

It is now easy to see that these bundles are locally isomorphic. Globally they may not be isomorphic, though. The reason is that while we may choose $T^*_{\mathbf{C}}X$ to be $T^*X \otimes \mathbf{C}$ globally, we do not know that this is true for $\Omega^1_{\mathcal{D}}(\mathcal{A})$. Nonetheless, up to a possible $U(1)$ twisting, they are globally isomorphic. It is easy to show using our change of coordinate functions f^i_{jk} that up to this possible phase factor the two bundles have the same transition functions. For the next step of the proof we require only local information, so this will not affect us. Later we will use the real structure to show that $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is actually untwisted.

From the above comments, we may easily deduce that

$$\Lambda^*_{\mathcal{D}}(\mathcal{A})|_{U^i} \cong \Gamma(\Lambda^*_{\mathbf{C}}(T^*X))|_{U^i}. \tag{124}$$

The action of $d = [\mathcal{D},\cdot]$ on this bundle may be locally determined, since we know that $\Lambda^*_{\mathcal{D}}(\mathcal{A})$ is a skew-symmetric graded differential algebra for d. First $d^2 = 0$, and d satisfies a graded Liebnitz rule on $\Lambda^*_{\mathcal{D}}(\mathcal{A})$. Furthermore, from the above local isomorphisms, given $f \in \mathcal{A}$,

$$df|_{U^i} = \sum_{j=1}^{p} \frac{\partial f}{\partial a^i_j}[\mathcal{D},a^i_j] = \sum_{j=1}^{p} \frac{\partial f}{\partial a^i_j} da^i_j. \tag{125}$$

By the uniqueness of the exterior derivative, characterised by these three properties, $[\mathcal{D},\cdot]$ is the exterior derivative on forms. We shall continue to write d or $[\mathcal{D},\cdot]$ as convenient.

Let us choose a connection compatible with the form $(\cdot,\cdot)_S$

$$\nabla : \mathcal{H}_\infty \to \Lambda^1_{\mathcal{D}}(\mathcal{A}) \otimes \mathcal{H}_\infty \tag{126}$$

$$\nabla(a\xi) = [\mathcal{D},a] \otimes \xi + a\nabla\xi. \tag{127}$$

Note that from the above discussion, this notion of connection agrees with our usual idea of covariant derivative. Denote by c the obvious map

$$c : End(\mathcal{H}_\infty) \otimes \mathcal{H}_\infty \to \mathcal{H}_\infty \tag{128}$$

and consider the composite map $c \circ \nabla : \mathcal{H}_\infty \to \mathcal{H}_\infty$. We have

$$\begin{aligned}(c \circ \nabla)(a\xi) &= [\mathcal{D},a]\xi + c(a\nabla\xi), \quad \forall a \in \mathcal{A},\ \xi \in \mathcal{H}_\infty \\ &= [\mathcal{D},a]\xi + ac(\nabla\xi)\end{aligned} \tag{129}$$

whereas

$$\mathcal{D}(a\xi) = [\mathcal{D}, a]\xi + a\mathcal{D}\xi \quad \forall a \in \mathcal{A},\ \xi \in \mathcal{H}_\infty. \tag{130}$$

Hence, on $\mathcal{H}_\infty$,

$$(c \circ \nabla - \mathcal{D})(a\xi) = a(c \circ \nabla - \mathcal{D})\xi \tag{131}$$

so that $c \circ \nabla - \mathcal{D}$ is $\mathcal{A}$-linear, or better, in the commutatant of $\mathcal{A}$. Thus if $c \circ \nabla - \mathcal{D}$ is bounded, it is in the weak closure of $\Omega^*_{\mathcal{D}}(\mathcal{A})$. However, as $(c \circ \nabla - \mathcal{D})\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$, it must in fact be in $\Omega^*_{\mathcal{D}}(\mathcal{A})$. The point of these observations is that if $c \circ \nabla - \mathcal{D}$ is bounded, then as ∇ is a first order differential operator (in particular having terms of integral order only) so is $\mathcal{D}$ (as elements of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ act as endomorphisms of $\mathcal{H}_\infty$, and so are order zero operators.) So let us show that $c \circ \nabla - \mathcal{D}$ is bounded. We know $\mathcal{H}_\infty \cong e\mathcal{A}^N$ for some N and $e \in M_N(\mathcal{A})$. As both $\mathcal{D}$ and ∇ have commutators with e in $\Omega^*_{\mathcal{D}}(\mathcal{A})$ (because $\mathcal{D}$, $c \circ \nabla : \mathcal{H}_\infty \to \mathcal{H}_\infty$) there is no loss of generality in setting e to 1 for our immediate purposes. So, simply consider the canonical generating set of $\mathcal{H}_\infty$ over $\mathcal{A}$ given by $\xi_j = (0, ..., 1, ..., 0)$, $j = 1, ..., N$. Then, there are $b_i^j, c_i^j \in \mathcal{A}$ such that

$$c \circ \nabla\xi_i = \sum_j b_i^j \xi_j, \quad \mathcal{D}\xi_i = \sum_j c_i^j \xi_j. \tag{132}$$

As $c \circ \nabla - \mathcal{D}$ is $\mathcal{A}$-linear, this shows that $c \circ \nabla - \mathcal{D}$ is bounded. Hence $\mathcal{D}$ is a first order differential operator. As the difference $c \circ \nabla - \mathcal{D}$ is in $\Omega^*_{\mathcal{D}}(\mathcal{A})$, $c \circ \nabla - \mathcal{D} = A$, for some element of $\Omega^*_{\mathcal{D}}(\mathcal{A})$. However, as $c \circ \nabla = \mathcal{D} + A$ is a connection (ignoring c), $A \in \Omega^1_{\mathcal{D}}(\mathcal{A})$.
Thus over U^i, we may write the matrix form of $\mathcal{D}$ as

$$\mathcal{D}^k{}_m = \sum_{j=1}^p \alpha_j{}^k{}_m \frac{\partial}{\partial a_j} + \beta^k{}_m \tag{133}$$

where $\beta^k{}_m$, $\alpha_j{}^k{}_m$ are bounded for each k, m. Similarly we write the square of $\mathcal{D}$ as

$$(\mathcal{D}^2)^n{}_m = \sum_{j,k} A_{jk}{}^n{}_m \frac{\partial^2}{\partial a_j \partial a_k} + \sum_k B_k{}^n{}_m \frac{\partial}{\partial a_k} + C^n{}_m \tag{134}$$

with all the terms A, B, C bounded, so that (as a pseudodifferential operator)

$$|\mathcal{D}|^n{}_m = \sum_k E_k{}^n{}_m \frac{\partial}{\partial a_k} + F^n{}_m \tag{135}$$

where E, F are bounded and

$$\sum_m E_k{}^n{}_m E_j{}^m{}_p = A_{kj}{}^n{}_p \tag{136}$$

et cetera. We will now show that the boundedness of $[|\mathcal{D}|, [\mathcal{D}, a]]$, required by the axioms, tells us that the first order part of $|\mathcal{D}|$ has a coefficient of the form fId_N, for some $f \in \mathcal{A}$. With the above notation,

$$[|\mathcal{D}|, [\mathcal{D}, a]]^n{}_p = \sum_{k,m} E_k{}^n{}_m \left(\frac{\partial[\mathcal{D}, a]^m{}_p}{\partial a_k}\right) + \sum_{k,m} (E_k{}^n{}_m [\mathcal{D}, a]^m{}_p - [\mathcal{D}, a]^n{}_m E_k{}^m{}_p) \frac{\partial}{\partial a_k} + [F, [\mathcal{D}, a]]^n{}_p. \tag{137}$$

For this to be bounded, it is necessary and sufficient that $[E_k, [\mathcal{D}, a_j]] = 0$, for all $j, k = 1, ..., p$. As

$$[|\mathcal{D}|, [\mathcal{D}, a_j][\mathcal{D}, a_k]] = [\mathcal{D}, a_j][|\mathcal{D}|, [\mathcal{D}, a_k]] + [|\mathcal{D}|, [\mathcal{D}, a_j]][\mathcal{D}, a_k], \tag{138}$$

and the commutant of $\Omega^*_{\mathcal{D}}(\mathcal{A})$ restricted to U^i is the weak closure of $\mathcal{A}$ restricted to U^i, the matrix E_k must be scalar over $\mathcal{A}$ for each k (not $\mathcal{A}''$ since $|\mathcal{D}|\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$). Thus $E_k{}^n{}_m = f_k\delta^n{}_m$, for some $f_k \in \mathcal{A}$. Since

$$A_{kj}{}^n{}_p = f_kf_j\delta^n{}_p \tag{139}$$

the leading order terms of $\mathcal{D}^2$ also have scalar coefficients.

Using the first order condition we see that

$$[\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j] = [[\mathcal{D}^2, a_j], a_k] = [[\mathcal{D}^2, a_k], a_j], \tag{140}$$

and denoting by $g^i_{jk} := ([\mathcal{D}, a^i_j], [\mathcal{D}, a^i_k])_{\Omega^1}$, we have

$$\frac{1}{p}Tr([\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j]) = -2Re(g^i_{jk}), \tag{141}$$

since $[\mathcal{D}, a_j]^* = -[\mathcal{D}, a_j]$. Now (140) is junk (since it is a graded commutator), and we are interested in the exact form of the right hand side. This is easily computed in terms of our established notation, and is given by

$$A_{jk} + A_{kj} = 2f_kf_jId_N. \tag{142}$$

Taken together, we have shown that

$$\begin{aligned}[\mathcal{D}, a_j][\mathcal{D}, a_k] + [\mathcal{D}, a_k][\mathcal{D}, a_j] &= [[\mathcal{D}^2, a_k], a_j]\\ &= A_{kj} + A_{jk}\\ &= 2f_kf_jId_N\\ &= -2Re(g^i_{jk})Id_N.\end{aligned} \tag{143}$$

This proves that

1) The $[\mathcal{D}, a^i_j]$ locally generate $Cliff(\Omega^1_{\mathcal{D}}(a^i_1, ..., a^i_p), Re(g^i_{jk}))$, by the universality of the Clifford relations. Also, from the form of the Hermitian structure on $\Omega^1_{\mathcal{D}}(\mathcal{A})$, $Re(g^i_{jk})$ is a nondegenerate quadratic form.

2) The operator $\mathcal{D}^2$ is a generalised Laplacian, as $f_kf_j = -Re(g^i_{jk})$.

3) From 2), the principal symbols $\sigma^{\mathcal{D}^2}_2(x, \xi) =\parallel \xi \parallel^2 Id$, $\sigma^{|\mathcal{D}|}_1(x, \xi) =\parallel \xi \parallel Id$, for $(x, \xi) \in T^*X|_{U^i}$, the total space of the cotangent bundle over U^i. This tells us that $|\mathcal{D}|$, $\mathcal{D}^2$ and $\mathcal{D}$ are elliptic differential operators, at least when restricted to the sets U^i. With a very little more work one can also see that $\sigma^{\mathcal{D}}_1(x, \xi) = \xi\cdot$, Clifford multiplication by ξ.

4) As $\Omega^*_{\mathcal{D}}(\mathcal{A})|_{U^i} \cong \mathbf{C}liff(T^*X)|_{U^i}$, and $\mathcal{H}_\infty$ is an irreducible module for $\Omega^*_{\mathcal{D}}(\mathcal{A})$, we see that S is the (unique) fundamental spinor bundle for X; see [21, appendix].

5) $\mathcal{D} = c \circ \nabla + A$, where ∇ is a compatible connection on the spinor bundle, and A is a self-adjoint element of $\Omega^1_{\mathcal{D}}(\mathcal{A})$. (Using the above results one can now show that $c \circ \nabla$ is essentially self-adjoint, whence A must be self-adjoint.)

6) It is possible to check that the connection on $\Lambda^*_{\mathcal{D}}(\mathcal{A}) \otimes \Gamma(X, S)$ given by the graded commutator $[\nabla, \cdot]$ is compatible with $(\cdot, \cdot)_{\Omega^1_{\mathcal{D}}}$. Hence ∇ is the lift of a compatible connection on the cotangent bundle.

The existence of an irreducible representation of $\mathbf{C}liff(T^*X \otimes \mathcal{L})$ for some line bundle $\mathcal{L}$ shows that X is a spin^c manifold. Before completing the proof that X is in fact spin, we briefly examine the metric.

It is now some time since Connes proved that his 'sup' definition of the metric coincided with the geodesic distance for the canonical triple on a spin manifold, [22]. We will reproduce the proof here for completeness. All one needs to know in order to show that these metrics agree is that for $a \in \mathcal{A}$ the operator $[\mathcal{D}, a] = \sum_j (\partial a/\partial a_j)[\mathcal{D}, a_j]$ is (locally, so over U^i for each i) Clifford multiplication by the gradient. Then Connes' proof holds with no modification:

$$\begin{aligned} \| [\mathcal{D}, a] \| &= \sup_{x\in X} |\sum_{j,k}(\frac{\partial a}{\partial a_j}[\mathcal{D}, a_j])^* \frac{\partial a}{\partial a_k}[\mathcal{D}, a_k]|^{1/2} \\ &= \sup_{x\in X} |\sum_{j,k} g^i_{jk}(\frac{\partial a}{\partial a_j})^* \frac{\partial a}{\partial a_k}|^{1/2} \\ &= \| a \|_{Lip} := \sup_{x\neq y} \frac{|a(x)-a(y)|}{d_\gamma(x,y)}. \end{aligned} \tag{144}$$

In the last line we have defined the Lipschitz norm, with $d_\gamma(\cdot,\cdot)$ the geodesic distance on X. The constraint $\| [\mathcal{D}, a] \| \leq 1$ forces $|a(x) - a(y)| \leq d_\gamma(x, y)$. To reverse the inequality, we fix x and observe that $d_\gamma(x,\cdot) : X \to \mathbf{R}$ satisfies $\| [\mathcal{D}, d_\gamma(x,\cdot)] \| \leq 1$. Then

$$\sup\{|a(x) - a(y)| : \| [\mathcal{D}, a] \| \leq 1\} = d(x, y) \geq |d_\gamma(x, y) - d_\gamma(x, x)| = d_\gamma(x, y). \tag{145}$$

Thus the two metrics $d(\cdot,\cdot)$ and $d_\gamma(\cdot,\cdot)$ agree.

4.3.5 X is Spin

In discussing the reality condition, we will need to recall that $Cliff_{r,s}$ module multiplication is, [21],

1) $\mathbf{R}$-linear for $r - s \equiv 0, 6, 7 \mod 8$
2) $\mathbf{C}$-linear for $r - s \equiv 1, 5 \mod 8$
3) $\mathbf{H}$-linear for $r - s \equiv 2, 3, 4 \mod 8$.

To show that X is spin, we need to show that there exists a representation of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$, where $\mathcal{A}_{\mathbf{R}} = \{a \in \mathcal{A} : a = a^*\}$. This is a real algebra with trivial involution. We will employ the properties of the real structure to do this, also extending the treatment to cover representations of $Cliff_{r,s}$, with $r + s = p$. This requires some background on Real Clifford algebras, [21, 23].

Let $Cliff(\mathbf{R}^{r,s})$ be the Real Clifford algebra on $\mathbf{R}^r \oplus \mathbf{R}^s$ with positive definite quadratic form and involution generated by $c : (x_1, ..., x_r, y_1, ..., y_s) \to (x_1, ..., x_r, -y_1, ..., -y_s)$ for $(x, y) \in \mathbf{R}^r \oplus \mathbf{R}^s$. The map c has a unique antilinear extension to the complexification $\mathbf{C}liff(\mathbf{R}^{r,s}) = Cliff(\mathbf{R}^{r,s}) \otimes \mathbf{C}$ given by $c \otimes cc$, where cc is complex conjugation. Note that all the algebras $Cliff_{r,s}$ with $r + s$ the same will become isomorphic when complexified, however this is not the case for the algebras $Cliff(\mathbf{R}^{r,s})$ with the involution. If we forget the involution, or if it is trivial, then $Cliff(\mathbf{R}^{r,s}) \cong Cliff_{r+s}$ and $\mathbf{C}liff(\mathbf{R}^{r,s}) \cong \mathbf{C}liff_{r+s}$.

A Real module for $Cliff(\mathbf{R}^{r,s})$ is a complex representation space for $Cliff_{r,s}$, W, along with an antilinear map (also called c) $c : W \to W$ such that

$$c(\phi w) = c(\phi)c(w) \quad \forall\phi \in Cliff(\mathbf{R}^{r,s}),\ \forall w \in W. \tag{146}$$

It can be shown, [21], that the Grothendieck group of Real representations of $Cliff(\mathbf{R}^{r,s})$ is isomorphic to the Grothendieck group of real representations of $Cliff_{r,s}$, and as every Real representation of $Cliff(\mathbf{R}^{r,s})$ automatically extends to $\mathbf{C}liff(\mathbf{R}^{r,s})$, the latter is the

appropriate complexification of the algebras $Cliff_{r,s}$. It also shows that KR-theory is the correct cohomological tool.

Pursuing the KR theme a little longer, we note that $(1,1)$-periodicity in this theory corresponds to the $(1,1)$-periodicity in the Clifford algebras

$$Cliff_{r,s} \cong Cliff_{r-s,0} \otimes Cliff_{1,1} \otimes \cdots \otimes Cliff_{1,1} \tag{147}$$

where there are s copies of $Cliff_{1,1}$ on the right hand side. As $Cliff_{1,1} \cong M_2(\mathbf{R})$ is a real algebra (as well as Real), this shows why the $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$-linearity of the module multiplication depends only on $r-s$ mod 8. We take $Cliff_{1,1}$ to be generated by 1_2 and $v=(v_1,v_2) \in \mathbf{R}^2$ by setting

$$v = \begin{pmatrix} v_2 & v_1 \\ -v_1 & -v_2 \end{pmatrix}$$

and the multiplication is just matrix multiplication

$$\begin{aligned} v \cdot w &= (v_2w_1 - v_1w_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - (v_1w_1 - v_2w_2)1_2 \\ &= v \wedge w \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - (v,w)_{1,1}1_2. \end{aligned} \tag{148}$$

We take $Cliff(\mathbf{R}^{1,1})$ to be generated by (v_1, iv_2) and we see that the involution is then given by complex conjugation. The multiplication is matrix multiplication with

$$v \cdot w = -(v,w)_2 1_2 + v \wedge w \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{149}$$

Thus we may always regard the involution on

$$\mathbf{C}liff(\mathbf{R}^{r,s}) \cong Cliff_{r-s,0} \otimes Cliff(\mathbf{R}^{1,1}) \otimes \cdots \otimes Cliff(\mathbf{R}^{1,1}) \otimes \mathbf{C} \tag{150}$$

as $1 \otimes cc \otimes cc \cdots \otimes cc \otimes cc$. This is enough of generalities for the moment. In our case we have a complex representation space $\Gamma(X,S)$, and an involution J such that

$$J(\phi\xi) = (J\phi J^*)(J\xi), \quad \forall \phi \in \Omega^*_{\mathcal{D}}(\mathcal{A}), \quad \forall \xi \in \Gamma(X,S). \tag{151}$$

So we actually have a representation of $\mathbf{C}liff(\mathbf{R}^{r,s})$ with the involution on the algebra realised by $J \cdot J^*$. It is clear that $J \cdot J^*$ has square 1, and so is an involution, and we set $s =$ number of eigenvalues equal to -1. Then from the preceeding discussion it is clear that

$$J \cdot J^* = 1_{Cliff_{p-2s,0}} \otimes cc \otimes \cdots \otimes cc \otimes cc \tag{152}$$

with s copies of cc acting on s copies of $Cliff(\mathbf{R}^{1,1})$ and with the behaviour of $J|_{Cliff_{p-2s,0}}$ determined by $p-2s$ mod 8 according to table (51). It is clear that $J \cdot J^*$ reduces to 1 on the positive definite part of the algebra, as it is an involution with all eigenvalues 1 there. This implies that $J \cdot J^*$ preserves elements of the form $\phi \otimes 1 \otimes \cdots \otimes 1 \otimes 1_{\mathbf{C}}$ where $\phi \in Cliff_{p-2s,0}$. However, we still need to fix the behaviour of J, and this is what is determined by $p-2s$ mod 8.

So we claim that we have a representation of $Cliff_{p-s,s}(T^*X, (J \cdot J^*, \cdot)_{\Omega^1})$ provided the behaviour of J is determined by $p-2s$ mod 8 and table 51. Two points: First, this reduces to

Connes' formulation for $s = 0$; second, the metric $(J \cdot J^*, \cdot)_{\Omega^1}$ has signature $(p-s, s)$ and making this adjustment corresponds to swapping between the multiplication on $Cliff(\mathbf{R}^{1,1})$ and $Cliff_{1,1}$. Similarly we replace $(\cdot,\cdot)_S$ with $(J\cdot,\cdot)_S$.

In all the above we have assumed that $2s \le p$. If this is not the case, we may start with the negative definite Clifford algebra, $Cliff_{0,2s-p}$, and then tensor on copies of $Cliff_{1,1}$.

Note that it is sufficient to prove the reduction for $0 < p \le 8$ and $s = 0$. This is because the extension to $s \neq 0$ involves tensoring on copies of $Cliff(\mathbf{R}^{1,1})$ for which the involution is determined, whilst raising the dimension simply involves tensoring on a copy of $Cliff_8 = M_{16}(\mathbf{R})$, and this will not affect the following argument. These simplifications reduce us to the case $J \cdot J^* = 1 \otimes cc$ on $\Omega^*_{\mathcal{D}}(\mathcal{A})$. To complete the proof, we proceed by cases.

The first case is $p = 6, 7, 8$. As $J^2 = 1$ and $J\mathcal{D} = \mathcal{D}J$, $J = cc$. We set $\Gamma_{\mathbf{R}}(X, S)$ to be the fixed point set of J. Then restricting to the action of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ on $\Gamma_{\mathbf{R}}(X, S)$, J is trivial. Hence we may regard the representation π as arising as the complexification of this real representation. As $\phi = J\phi = \phi J = J\phi J^*$ on $\Gamma_{\mathbf{R}}(X, S)$, the action can only be $\mathbf{R}$-linear. From the fact that $[\mathcal{D}, J] = 0$, we easily deduce that $\nabla J = 0$, so that J is globally parallel. Thus there is no global twisting involved in obtaining $\Omega^*_{\mathcal{D}}(\mathcal{A})$ from $Cliff(T^*X)$. Hence X is spin.

In dimensions $2, 3, 4$, not only does J commute with $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$, but i does also (we are looking at the action on $\Gamma(X, S)$, not $\Gamma_{\mathbf{R}}(X, S)$). So set

$$e = J, \quad f = i, \quad g = Ji, \tag{153}$$

note that $e^2 = f^2 = g^2 = -1$, and observe that the following commutation relations hold:

$$ef = -fe = g, \quad fg = -gf = e, \quad ge = -eg = f. \tag{154}$$

Thus regarding e, f, g and $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ as elements of $\hom_{\mathbf{R}}(\Gamma(X, S), \Gamma(X, S))$, we see that $\Gamma(X, S)$ has the structure of a quaternion vector bundle on X, and the action of $Cliff(T^*X)$ is quaternion linear. As in the last case, $\nabla J = 0$, so that the Clifford bundle is untwisted and so X is spin.

The last case is $p = 1, 5$. For $p = 1$, the fibres of $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ are isomorphic to $\mathbf{C}$, and we naturally have that the Clifford multiplication is $\mathbf{C}$-linear. For $p = 5$, the fibres are $M_4(\mathbf{C})$, and as $J^2 = -1$, we have a commuting subalgebra $span_{\mathbf{R}}\{1, J\} \cong \mathbf{C}$. Note that the reason for the anticommutation of J and $\mathcal{D}$ is that $\mathcal{D}$ maps real functions to imaginary functions, for $p = 1$, and so has a factor of i. Analogous statements hold for $p = 5$. In particular, removing the complex coefficients, so passing from $\mathcal{D}$ to ∇, we see that $\nabla J = 0$, and so X is spin.

Note that in the even dimensional cases when $\pi(c)J = J\pi(c)$, $\pi(c) \in \Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$. When they anticommute, $\pi(c)$ is i times a real form. This corresponds to the behaviour of the complex volume form of a spin manifold on the spinor bundle. Compare the above discussion with [21].

It is interesting to consider whether we can recover the indefinite distance from $(J \cdot J^*, \cdot)_{\Omega^1}$. We will not address the issue here, but simply point out that in the topology determined by $(J \cdot J^*, \cdot)$, our previously compact space is no longer necessarily compact, and so can not agree with the weak* topology. It is worth noting that if $J \cdot J^*$ has one or more negative eigenvalues and ∇ is compatible with the Hermitian form $(J\cdot,\cdot)_S$, then $\mathcal{D} = c \circ \nabla$ is hyperbolic rather than elliptic. So many remaining points of the proof, relying on the ellipticity of $\mathcal{D}$, will not go through for the pseudo-Riemannian case. We will however point out the occasional interesting detail for this case.

So for all dimensions we have shown that X is a spin manifold with $\mathcal{A}$ the smooth functions on X acting as multiplication operators on an irreducible spinor bundle. Thus 3) is proved completely.

4.4 Completion of the Proof.

4.4.1 Generalities and Proof of 4).

To prove 4), note that if we make a unitary change of representation, the metric, the integration defined via the Dixmier trace, and the absolutely continuous spectrum of the a^i_j (i.e. X), are all unchanged. The only object in sight that varies in any important way with unitary change of representation is the operator $\mathcal{D}$. The change of representation induces an affine change on $\mathcal{D}$:

$$\mathcal{D} \to U\mathcal{D}U^* = \mathcal{D} + U[\mathcal{D}, U^*]. \tag{155}$$

This in itself shows that the connected components of the fibre over $[\pi] \to d_\pi(\cdot,\cdot)$ are affine. To show that there are a finite number of components, it suffices to note that a representation in any component satisfies the axioms, (recall that a spin structure for one metric canonically determines one for any other metric, [21]), and so gives rise to an action of the Clifford bundle, and so to a spin structure. As there are only a finite number of these, we have proved 4).

The only items remaining to be proved are, for $p > 2$,

1. $⨍ |\mathcal{D}|^{2-p}$ is a positive definite quadratic form on each A_σ with unique minimum π_σ
2. This minimum is achieved for $\mathcal{D} = \not{D}$, the Dirac operator on S_σ
3. $⨍ |\not{D}|^{2-p} = -\frac{(p-2)c(p)}{12}\int_X R dv$.

These last few items will all be proved by direct computation once we have narrowed down the nature of $\mathcal{D}$ a bit more. As an extra bonus, we will also be able to determine the measure once we have this extra information.

Recall the condition for compatibility of a connection ∇^S on S with the Hermitian structure $(\cdot,\cdot)_S$ as

$$[\mathcal{D}, (\xi,\eta)_S] = (\xi, \nabla^S\eta)_S - (\nabla^S\xi, \eta)_S, \quad \forall \xi, \eta \in \Gamma(X, S). \tag{156}$$

Given such a connection, the graded commutator $[\nabla^S, \cdot] : \Lambda^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}}) \to \Lambda^{*+1}_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$ is a connection compatible with the metric on $\Lambda^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}})$. If instead we have a connection compatible with $(J\cdot,\cdot)_S$, then $[\nabla^S,\cdot]$ is compatible with $(J\cdot J^*,\cdot)_{\Omega^1_{\mathcal{D}}}$. Note that we are really considering differential forms with values in $\Gamma(X,S)$, so actually have a connection $[\nabla^S,\cdot] : \Lambda^*_{\mathcal{D}}(\mathcal{A}) \otimes \Gamma(X,S) \to \Lambda^{*+1}_{\mathcal{D}}(\mathcal{A}) \otimes \Gamma(X,S)$. Beware of confusing the notation here, for $[\nabla^S,\cdot]$ uses the graded commutator, while $[\mathcal{D},\cdot](a[\mathcal{D},b]) = [\mathcal{D},a][\mathcal{D},b]$.

The torsion of the connection $[\nabla^S,\cdot]$ on T^*X is defined to be $T([\nabla^S,\cdot]) = d - \epsilon \circ [\nabla^S,\cdot]$, where $d = [\mathcal{D},\cdot]$ and ϵ is just antisymmetrisation. Then from what has been proved thus far, we have

$$\mathcal{D} = c \circ \nabla^S + T, \quad [\mathcal{D},\cdot] = c \circ [\nabla^S,\cdot] + c \circ T([\nabla^S,\cdot]), \tag{157}$$

on $\Gamma(X,S)$ and $\Omega^*_{\mathcal{D}}(\mathcal{A}_{\mathbf{R}}) \otimes \Gamma(X,S)$ respectively. Here c is the composition of Clifford multiplication with the derivation in question. On the bundle $\Lambda^*_{\mathcal{D}}(\mathcal{A}) \otimes \Gamma(X,S)$ we have already seen that $[\mathcal{D},\cdot]$ is the exterior derivative. The T in the expression for $\mathcal{D}$ is the lift of the torsion term to the spinor bundle.

Any two compatible connections on S differ by a 1−form, A say, and by virtue of the first order condition, adding A to ∇^S does not affect $[\nabla^S,\cdot]$, and so in particular ∇^S would still be the lift of a compatible connection on the cotangent bundle. As $U[\mathcal{D},U^*]$ is self-adjoint, for

any representation π, the operator D_π is the Dirac operator of a compatible connection on the spinor bundle. Note that as $\mathcal{D}$ is self-adjoint, the Clifford action of any such 1-form A must be self-adjoint on the spinor bundle.

It is important to note that for every unitary element of the algebra, u say, gives rise to a unitary transformation $U = uJuJ^*$. If we start with $\mathcal{D} + A$, $A = A^* \in \Omega^1_{\mathcal{D}}(\mathcal{A})$, and conjugate by U, we obtain $\mathcal{D} + A + JAJ^* + u[\mathcal{D}, u^*] + Ju[\mathcal{D}, u^*]J^*$. If the metric is positive definite, then $JAJ^* = -A^*$ for all $A \in \Omega^*_{\mathcal{D}}(\mathcal{A})$. Thus all of these gauge terms (or internal fluctuations, [1]) vanish in the positive definite, commutative case. This corresponds to the Clifford algebra being built on the untwisted cotangent bundle, so that we do not have any $U(1)$ gauge terms. In the indefinite case we find non-trivial gauge terms associated with timelike directions. To see this, note that every element of $\Omega^1_{\mathcal{D}}(\mathcal{A})$ is of the form $A + iB$, where each of A and B are real, so anti-self-adjoint. Possible gauge terms are of the form iB, as they must be self-adjoint. If we assume that B is timelike (i.e. $JBJ^* = -B$), and set $(u[\mathcal{D}, u^*])_t$ to be the timelike part of $u[\mathcal{D}, u^*]$, then

$$\begin{aligned} U(\mathcal{D} + iB)U^* &= \mathcal{D} + iB + JiBJ^* + u[\mathcal{D}, u^*] + Ju[\mathcal{D}, u^*]J^* \\ &= \mathcal{D} + iB - iJBJ^* + u[\mathcal{D}, u^*] + Ju[\mathcal{D}, u^*]J^* \\ &= \mathcal{D} + 2iB + 2(u[\mathcal{D}, u^*])_t. \end{aligned} \tag{158}$$

Thus we can find non-trivial gauge terms in timelike directions.

Since we are unequivocably in the manifold setting now, and as we shall require the symbol calculus to compute the Wodzicki residue, we shall now change notation. In traditional fashion, let us write

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}1_S \tag{159}$$

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}1_S \tag{160}$$

for the curved (coordinate) and flat (orthonormal) gamma matrices respectively. Let σ^k, $k = 1, ..., [p/2]$, be a local orthonormal basis of $\Gamma(X, S)$, and $a \in \pi(\mathcal{A})$. Then the most general form that $\mathcal{D}_\pi$ can take is

$$\mathcal{D}_\pi(a\sigma^k) = \sum_\mu \gamma^\mu(\partial_\mu a)\sigma^k + \frac{1}{2}\sum_{\mu,a<b} a\gamma^\mu\omega_{\mu ab}\gamma^a\gamma^b\sigma^k + \frac{1}{2}\sum_{\mu,a<b} a\gamma^\mu t_{\mu ab}\gamma^a\gamma^b\sigma^k + a\sum_\mu \gamma^\mu f_\mu\sigma^k \tag{161}$$

where ω is the lift of the Levi-Civita connection to the bundle of spinors, t is the lift of the torsion term, and f_μ is a gauge term associated to timelike directions. We assume without loss of generality that our coordinates allow us to split the cotangent space so that timelike and spacelike terms are orthogonal. Then we may take $f_\mu = 0$ for μ the index of a spacelike direction. We will now drop the π and consider $\mathcal{D}$ as being determined by t and f_μ. It is worth noting that t_{cab} is totally antisymmetric, where $t_{\mu ab} = e^c_\mu t_{cab}$, and e^c_μ is the vielbein. Note that from our previous discussion, the appropriate choice of Dirac operator is no longer elliptic, and so in the following arguments we will assume that $J \cdot J^*$ has no negative eigenvalues. Thus from this point on we assume that we are in the positive definite case with $f_\mu = 0$ and $\mathcal{D} = \mathcal{D}(t)$.

This gives us enough information to recover the measure on our space also. All of these operators, $\mathcal{D}(t)$, have the same principal symbol, $\xi\cdot$, Clifford multiplication by ξ. Hence, over the unit sphere bundle the principal symbol of $|\mathcal{D}|$ is 1. Likewise, the restriction of the principal symbol of $a|\mathcal{D}|^{-p}$ to the unit sphere bundle is a, where here we mean $\pi(a)$, of course. Before evaluating the Dixmier trace of $a|\mathcal{D}|^{-p}$, let us look at the volume form.

Since the $[\mathcal{D}, a_j^i]$ are independent at each point of U^i, the sections $[\mathcal{D}, a_j^i]$, $j = 1, ..., p$, form a (coordinate) basis of the cotangent bundle. Then their product is the real volume form ω^i. With $\omega_{\mathbf{C}} = i^{[(p+1)/2]}\omega$ the complex volume form, we have

$$\Gamma = \pi(c) = \sum_i a_0^i[\mathcal{D}, a_1^i] \cdots [\mathcal{D}, a_p^i] = \sum_i \tilde{a}_0^i \omega_{\mathbf{C}}^i \tag{162}$$

where $a_0^i = \tilde{a}_0^i i^{[(p+1)/2]}$, [21].

As $\omega_{\mathbf{C}}$ is central over U^i for p odd, it must be a scalar multiple, k, of the identity. So $\sum_i k\tilde{a}_0^i(x) = k$, and we see that the collection of maps $\{\tilde{a}_0^i\}_i$ form a partition of unity subordinate to the U^i. The axioms tell us that $k = 1$. In the even case, $\omega_{\mathbf{C}}$ gives the $\mathbf{Z}_2$-grading of the Hilbert space,

$$\mathcal{H} = \frac{1 + \omega_{\mathbf{C}}}{2}\mathcal{H} \oplus \frac{1 - \omega_{\mathbf{C}}}{2}\mathcal{H}. \tag{163}$$

This corresponds to the splitting of the spin bundle, and for sections of these subbundles we have

$$1 = \sum_i \tilde{a}_0^i \frac{1 + \omega_{\mathbf{C}}^i}{2} = \sum_i \tilde{a}_0^i \tag{164}$$

and similarly for $\frac{1-\omega_{\mathbf{C}}}{2}$. Thus in the even dimensional case we also have a partition of unity.

Recall the usual definition of the measure on X. To integrate a function $f \in \mathcal{A}$ over a single coordinate chart U^i, we make use of the (local) embedding $a^i : U^i \to \mathbf{R}^p$. We write $f = \tilde{f}(a_1^i, ..., a_p^i)$ where $\tilde{f} : \mathbf{R}^p \to \mathbf{C}$ has compact support. Then

$$\int_{U^i} f := \int_{U^i} (a^i)^*(\tilde{f}) = \int_{a^i(U^i)} \tilde{f}(x) d^p x. \tag{165}$$

To integrate f over X, we make use of the embedding a and the partition of unity and write

$$\sum_i \int_{a^i(U^i)} (\tilde{a}_0^i \tilde{f})(x) d^p x. \tag{166}$$

Now given a smooth space like X, a representation of the continuous functions will split into two pieces; one absolutely continuous with respect to the Lebesgue measure, above, and one singular with respect to it, [13], $\pi = \pi_{ac} \oplus \pi_s$. This gives us a decomposition of the Hilbert space into complementary closed subspaces, $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_s$. The joint spectral measure of the a_j^i, $j = 1, ..., p$, is absolutely continuous with respect to the p dimensional Lebesgue measure, so $\overline{\mathcal{H}_\infty} \subseteq \mathcal{H}_{ac}$. By the definition of the inner product on $\mathcal{H}_\infty$ given in the axiom of finiteness and absolute continuity, $\overline{\mathcal{H}_\infty} = L^2(X, S, ⨍ \cdot|\mathcal{D}|^{-p})$. As the Lebesgue measure on the joint absolutely continuous spectrum is itself absolutely continuous with respect to the measure given by the Dixmier trace, we must also have $\mathcal{H}_{ac} \subseteq \overline{\mathcal{H}_\infty}$, and so they are equal. As all the a_j^i act as zero on $\mathcal{H}_s$, recall they are smooth elements, and they generate both $\mathcal{A}$ and $\overline{\mathcal{A}}$, the requirement of irreducibilty says that $\mathcal{H}_s = 0$. Thus the representation is absolutely continuous, and as the measure is in the same measure class as the Lebesgue measure, $\mathcal{H} = \overline{\mathcal{H}_\infty} = L^2(X, S)$.

Let us now compute the value of the integral given by the Dixmier trace. From the form of $\mathcal{D}$, we know that $\mathcal{D}$ is an operator of order 1 on the spinor bundle of X, so $|\mathcal{D}|^{-p}$ is of order $-p$. Invoking Connes' trace theorem

$$\begin{aligned}\rlap{\,-}\!\!\int f|\mathcal{D}|^{-p} &= \frac{1}{p(2\pi)^p}\sum_i \int_{S^*U^i} tr_S(\tilde{a}_0^i f)\sqrt{g}d^px d\xi \\ &= \frac{2^{[p/2]}Vol(S^{p-1})}{p(2\pi)^p}\sum_i \int_{U^i} \tilde{a}_0^i f\sqrt{g}d^px. \end{aligned} \tag{167}$$

Thus the inner product on $\mathcal{H}$ is given by

$$\langle a\xi, \eta\rangle = \frac{2^{[p/2]}Vol(S^{p-1})}{p(2\pi)^p}\int_X (a^*(\xi,\eta)_S\sqrt{g})(x)d^px. \tag{168}$$

We note for future reference that $\mathrm{Vol}(S^{p-1}) = \frac{(4\pi)^{p/2}}{2^{p-1}\Gamma(p/2)}$, [26], so that the complete factor above is the same as in equation (65),

$$\frac{2^{[p/2]}Vol(S^{p-1})}{p(2\pi)^p} = c(p). \tag{169}$$

All the above discussion is limited to the case $p \neq 1$. The only 1 dimensional compact spin manifold is S^1. In this case the Dirac operator is $\frac{1}{i}\frac{d}{dx}$, with singular values $\mu_n(|\mathcal{D}|^{-1}) = \frac{1}{n}$. In [6, pp 311,312], Connes presents an argument bounding the $(p,1)$ norm of $[f(\epsilon\mathcal{D}), a]$ in terms of $[\mathcal{D}, a]$ and the Dixmier trace of $\mathcal{D}$, with $\epsilon > 0$ and f a smooth, even, compactly supported, real function. From this Theorem 7 is a consequence of specialising f. Our aim then is to bound the trace of $[f(\epsilon\mathcal{D}), a]$. So suppose that the support of f is contained in $[-k, k]$. Then the rank of $[f(\epsilon\mathcal{D}), a]$ is bounded by the number of eigenvalues of $|\mathcal{D}|^{-1} \geq \epsilon k^{-1}$. Calling this number N, we have $N \leq \epsilon^{-1}k$ and so

$$\| [f(\epsilon\mathcal{D}), a] \|_1 \leq 2\epsilon^{-1}k \| [f(\epsilon\mathcal{D}), a] \| . \tag{170}$$

The rest of the argument uses Fourier analysis techniques to bound the commutator in terms of $\| [\mathcal{D}, a] \|$, and noting that $\sum^N \frac{1}{n}/\log N \geq 1$, for then

$$\| [f(\epsilon\mathcal{D}), a] \|_1 \leq 2C_f \| [\mathcal{D}, a] \| \rlap{\,-}\!\!\int |\mathcal{D}|^{-1}. \tag{171}$$

From this a choice of f gives the analogue of Theorem 7 in this case, as the results of Voiculescu and Wodzicki hold for dimension 1; see [6] for the full story.

As the Dirac operator of a compatible Clifford connection is self-adjoint only when there is no boundary, the self-adjointness of $\mathcal{D}$ and the geometric interpretation of the inner product on the Hilbert space now shows that the spin manifold X is closed. There are numerous consequences of closedness, as well as a more general formulation for the noncommutative case; see [6]. All that remains is to examine the gravity action given by the Wodzicki residue.

4.4.2 The Even Dimensional Case.

Much of what follows is based on [24], though we also complete the odd dimensional case. We also note that this calculation was carried out in the four dimensional case in [25]. The key to the following computations is the composition formula for symbols:

$$\sigma(P\circ Q)(x,\xi) = \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha \sigma(P))(\partial_x^\alpha \sigma(Q)). \tag{172}$$

We shall use this to determine $\sigma_{-p}(|\mathcal{D}|^{2-p})$, so that we may compute the Wodzicki residue. In the even dimensional case, we use this formula to obtain the following,

$$\begin{aligned}\sigma_{-p}(\mathcal{D}^{2-p}) &= \sigma_0(\mathcal{D}^2)\sigma_{-p}(\mathcal{D}^{-p}) + \sigma_1(\mathcal{D}^2)\sigma_{-p-1}(\mathcal{D}^{-p}) \\ &+\sigma_2(\mathcal{D}^2)\sigma_{-p-2}(\mathcal{D}^{-p}) - i\sum_\mu(\partial_{\xi_\mu}\sigma_1(\mathcal{D}^2))(\partial_{x^\mu}\sigma_{-p}(\mathcal{D}^{-p})) \\ &-i\sum_\mu(\partial_{\xi_\mu}\sigma_2(\mathcal{D}^2))(\partial_{x^\mu}\sigma_{-p-1}(\mathcal{D}^{-p})) \\ &-\frac{1}{2}\sum_{\mu,\nu}(\partial^2_{\xi_\mu\xi_\nu}\sigma_2(\mathcal{D}^2))(\partial^2_{x^\mu x^\nu}\sigma_{-p}(\mathcal{D}^{-p})). \end{aligned} \tag{173}$$

This involves the symbol of $\mathcal{D}^2$ which we can compute, and lower order terms from $|\mathcal{D}|^{-p}$. Since $|\mathcal{D}|^2 = \mathcal{D}^2$, we have a simplification in the even dimensional case, namely that the expansion

$$\sigma(\mathcal{D}^{-2m}) = \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha\sigma(\mathcal{D}^{-2m+2})(\partial_x^\alpha\sigma(\mathcal{D}^{-2})), \tag{174}$$

provides a recursion relation for the lower order terms provided we can determine the first few terms of the symbol for a parametrix of $\mathcal{D}^2$. Let $\sigma_2 = \sigma_2(\mathcal{D}^2)$ and $p = 2m$. Then by the multiplicativity of principal symbols, or from the above, $\sigma_{-2m}(\mathcal{D}^{-2m}) = \sigma_2^{-m}$, at least away from the zero section. Also let us briefly recall that while principal symbols are coordinate independent, other terms are not. So all the following calculations will be made in Riemann normal coordinates, for which the metric takes the simplifying form

$$g^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{1}{3}R^{\mu\nu}_{\rho\sigma}(x_0)x^\rho x^\sigma + O(x^3). \tag{175}$$

This choice will simplify many expressions, and we will write $=_{RN}$ to denote equality in these coordinates. Also, as we will be interested in the value of certain expressions on the cosphere bundle, we will also employ the symbol $=_{RN,\ \mathrm{mod}\ \|\xi\|}$ to denote a Riemann normal expression in which $\|\ \xi\ \|$ has been set to 1. So using (174) to write

$$\begin{aligned}\sigma_{-2m-1}(\mathcal{D}^{-2m}) &= \sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2}) + \sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_2^{-1} \\ &-i\sum_\mu(\partial_{\xi_\mu}\sigma_2^{-m+1})(\partial_{x^\mu}\sigma_2^{-1}), \end{aligned} \tag{176}$$

we can use Riemann normal coordinates to simplify this to

$$\begin{aligned}\sigma_{-2m-1}(\mathcal{D}^{-2m}) &=_{RN} \sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2}) \\ &+\sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_2^{-1} \\ &=_{RN} m\sigma_2^{-m+1}\sigma_{-3}(\mathcal{D}^{-2}), \end{aligned} \tag{177}$$

after applying recursion in the obvious way. The next term to compute is

$$\begin{aligned}\sigma_{-2m-2}(\mathcal{D}^{-2m}) &= \sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) + \sigma_{-2m+1}(\mathcal{D}^{-2m+2})\sigma_{-3}(\mathcal{D}^{-2}) \\ &+\sigma_{-2m}(\mathcal{D}^{-2m+2})\sigma_2^{-1} - i\sum_\mu(\partial_{\xi_\mu}\sigma_2^{-m+2})(\partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2})) \\ &-\frac{1}{2}\sum_{\mu,\nu}(\partial^2_{\xi_\mu\xi_\nu}\sigma_2^{-m+1})(\partial^2_{x^\mu x^\nu}\sigma_2^{-1}). \end{aligned} \tag{178}$$

Using the last result and the following two expressions

$$\partial^2_{\xi_\mu \xi_\nu} \parallel \xi \parallel^{-2m+2} =_{RN} 2m(2m-2)\sigma_2^{-m-1}\delta^{\mu\tau}\xi_\tau\delta^{\nu\sigma}\xi_\sigma - (m-1)\sigma_2^{-m}\delta^{\mu\nu},$$

$$\partial^2_{x^\mu x^\nu} \parallel \xi \parallel^{-2} =_{RN} \frac{1}{3} R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma\sigma_2^{-2}$$

we find

$$\begin{aligned}\sigma_{-2m-2}(\mathcal{D}^{-2m}) =_{RN} &\ \sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) + (m-1)\sigma_2^{-m+2}(\sigma_{-3}(\mathcal{D}^{-2}))^2 \\ &+\sigma_{-2m}(\mathcal{D}^{-2m+2})\sigma_2^{-1} + 2i(m-1)\delta^{\mu\sigma}\xi_\sigma\partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2}) \\ &-\frac{4m(m-1)}{3}\sigma_2^{-m-3}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma + \frac{(m-1)}{3}\sigma_2^{-m-2}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_{rho}\xi_\sigma. \end{aligned} \tag{179}$$

Given $\sigma_{-4}(\mathcal{D}^{-2})$ and $\sigma_{-3}(\mathcal{D}^{-2})$ this can be computed recursively, giving

$$\begin{aligned}\sigma_{-2m-2}(\mathcal{D}^{-2m}) =_{RN} &\ m\sigma_2^{-m+1}\sigma_{-4}(\mathcal{D}^{-2}) + \frac{m(m-1)}{2}\sigma_2^{-m+2}(\sigma_{-3}(\mathcal{D}^{-2}))^2 \\ &+im(m-1)\xi^\mu\partial_{x^\mu}\sigma_{-3}(\mathcal{D}^{-2}) \\ &-\frac{4m(m+1)(m-1)}{9}\sigma_2^{-m-3}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma \\ &+\frac{m(m-1)}{6}\sigma_2^{-m-2}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma, \end{aligned} \tag{180}$$

where $\xi^\mu = \delta^{\mu\nu}\xi_\nu$. In the even case, this gives us a short cut; we shall compute this term in general for the odd case, but note that in the even case the short cut gives us

$$\begin{aligned}\sigma_{-p}(\mathcal{D}^{-p+2}) =_{RN,\ \mathrm{mod}\ \|\xi\|} &\ \frac{(p-2)}{2}\sigma_{-4}(\mathcal{D}^{-2}) + \frac{(p-2)(p-4)}{8}(\sigma_{-3}(\mathcal{D}^{-2}))^2 \\ &+\frac{(p-2)(p-4)}{4}i\xi^\mu\partial_\mu\sigma_{-3}(\mathcal{D}^{-2}) - \frac{p(p-2)(p-4)}{18}\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma \\ &+\frac{(p-2)(p-4)}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma. \end{aligned} \tag{181}$$

Having obtained $\sigma_{-2m-2}(\mathcal{D}^{-2})$ and $\sigma_{-2m-1}(\mathcal{D}^{-2})$, the next step is to compute $\sigma_{-3}(\mathcal{D}^{-2})$ and $\sigma_{-4}(\mathcal{D}^{-2})$. We follow the method of [24] to construct a parametrix for $\mathcal{D}^2$.
First, let us write $\mathcal{D}^2$ in elliptic operator form

$$\mathcal{D}^2 = -g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b. \tag{182}$$

So the symbol of $\mathcal{D}^2$ is

$$\begin{aligned}\sigma(\mathcal{D}^2) &= g^{\mu\nu}\xi_\mu\xi_\nu + ia^\mu\xi_\mu + b \\ &=\parallel \xi \parallel^2 + ia^\mu\xi_\mu + b \\ &= \sigma_2 + \sigma_1 + \sigma_0. \end{aligned} \tag{183}$$

With this notation in hand, let P be the pseudodifferential operator defined by $\sigma(P) = \sigma_2^{-1}$. In fact we should consider the product $\chi(|\xi|)\sigma_2(x,\xi)^{-1}$, where χ is a smooth function vanishing

for small values of its (positive) argument. As this does not affect the following argument, only altering the result by an infinitely smoothing operator, we shall omit further mention of this "mollifying function". So, one readily checks that $\sigma(\mathcal{D}^2 P - 1)$ is a symbol of order -1. Denoting this symbol by r, we have

$$\sigma(\mathcal{D}^2 P) = 1 + r \quad \text{so} \quad \sigma(\mathcal{D}^2 P) \circ (1+r)^{-1} \sim 1 \tag{184}$$

where on the right composition means the symbol of the composition of operators. So if $\sigma(R) = 1 + r$, then $\mathcal{D}^2 P R^{-1} \sim 1$. Hence $PR^{-1} \sim \mathcal{D}^{-2}$. As r is of order -1, we may expand $(1+r)^{-1}$ as a geometric series in symbol space. Thus

$$\begin{aligned}
\sigma(\mathcal{D}^{-2}) &\sim \sigma_2^{-1} \circ \sum_{k=0}^{\infty} (-1)^k r^{\circ k} \\
&\sim \sigma_2^{-1} \circ (1 - r + r^{\circ 2} - r^{\circ 3} + \ldots) \\
&\sim \sigma_2^{-1} - \sigma_2^{-1} \circ r + \sigma_2^{-1} \circ r \circ r + \text{order -5.}
\end{aligned} \tag{185}$$

It is straightforward to compute the part of order -1 of r

$$\begin{aligned}
r_{-1} &= i a^{\mu} \xi_{\mu} \sigma_2^{-1} + 2i \xi^{\mu} g^{\rho\tau}_{,\mu} \xi_{\rho} \xi_{\tau} \sigma_2^{-1} \\
&=_{RN} i a^{\mu} \xi_{\mu} \sigma_2^{-1}
\end{aligned} \tag{186}$$

and its derivative

$$\partial_{x^{\mu}} r_{-1} =_{RN} i a^{\rho}_{,\mu} \xi_{\rho} \sigma_2^{-1} - \frac{2i}{3} \xi^{\rho} R^{\alpha\tau}_{\rho\mu} \xi_{\alpha} \xi_{\tau} \sigma_2^{-2}, \tag{187}$$

as well as the part of order -2

$$r_{-2} =_{RN} b \sigma_2^{-1} - \frac{2}{3} \delta^{\mu\nu} R^{\rho\sigma}_{\mu\nu} \xi_{\rho} \xi_{\sigma} \sigma_2^{-2}. \tag{188}$$

Using the composition formula (repeatedly) and discarding terms of order -5 or less, we eventually find that

$$\sigma_{-3}(\mathcal{D}^{-2}) = -i a^{\mu} \xi_{\mu} \sigma_2^{-2}, \tag{189}$$

and

$$\begin{aligned}
\sigma_{-4}(\mathcal{D}^{-2}) = &-b \sigma_2^{-2} + \frac{2}{3} \delta^{\mu\nu} R^{\alpha\tau}_{\mu\nu} \xi_{\alpha} \xi_{\tau} \sigma_2^{-3} \\
&+ 2 \xi^{\mu} a^{\rho}_{,\mu} \xi_{\rho} \sigma_2^{-3} - a^{\mu} \xi_{\mu} a^{\rho} \xi_{\rho} \sigma_2^{-3} \\
&- \tfrac{4}{3} \xi^{\mu} \xi^{\nu} R^{\alpha\tau}_{\nu\mu} \xi_{\alpha} \xi_{\tau} \sigma_2^{-4}.
\end{aligned} \tag{190}$$

Employing the shortcut for the even case yields

$$\begin{aligned}
\sigma_{-p}(\mathcal{D}^{-p+2}) =_{RN,\ \mathrm{mod}\ \|\xi\|} &-\frac{1}{2}(p-2) b + \frac{p(p-2)}{4} \xi^{\mu} a^{\rho}_{,\mu} \xi_{\rho} \\
&- \frac{p(p-2)}{8} a^{\mu} \xi_{\mu} a^{\rho} \xi_{\rho} + \frac{p(p-2)}{24} \delta^{\mu\nu} R^{\rho\sigma}_{\mu\nu} \xi_{\rho} \xi_{\sigma} \\
&- \frac{(p-2)(p^2 - 4p + 6)}{18} \xi^{\mu} \xi^{\nu} R^{\rho\sigma}_{\mu\nu} \xi_{\rho} \xi_{\sigma}.
\end{aligned} \tag{191}$$

In order to perform the integral over the cosphere bundle, we make use of the standard results

$$\int_{\|\xi\|=1} \xi^\mu d\xi = 0, \quad \int_{\|\xi\|=1} \xi^\mu \xi^\nu \xi^\rho d\xi = 0, \quad \int_{\|\xi\|=1} \xi^\mu \xi^\nu d\xi = \frac{1}{p} g^{\mu\nu}, \tag{192}$$

and

$$\int_{\|\xi\|=1} \xi^\mu \xi^\nu \xi^\rho \xi^\sigma d\xi = \frac{1}{p(p+2)} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\sigma\nu} g^{\mu\rho}). \tag{193}$$

Using the symmetries of the Riemann tensor, one may use the last result to show that

$$\int_{S^*X} R^{\mu\nu}_{\alpha\beta} \xi^\alpha \xi^\beta \xi^\sigma \xi^\tau g_{\sigma\mu} g_{\tau\nu} d\xi dx = 0. \tag{194}$$

Thus

$$\begin{aligned} WRes(\mathcal{D}^{2-p}) &= \frac{1}{p(2\pi)^p} \int_{S^*X} tr\sigma_{-p}(\mathcal{D}^{2-p}) \sqrt{g} d\xi dx \\ &= -\frac{(p-2)}{2} \frac{Vol(S^{p-1})}{p(2\pi)^p} \int_X tr(b + \frac{1}{4} a^\mu a_\mu - \frac{1}{2} a^\mu_{,\mu}) \sqrt{g} dx \\ &+ \frac{(p-2)}{24} \frac{Vol(S^{p-1}) 2^{[p/2]}}{p(2\pi)^p} \int_X R \sqrt{g} dx. \end{aligned} \tag{195}$$

To make use of this we will need expressions for a^μ and b. The art of squaring Dirac operators is well described in the literature, and we follow [24]. Writing

$$\mathcal{D} = \gamma^\mu (\nabla_\mu + T_\mu) \tag{196}$$

the square may be written, with ∇ the lift of the Levi-Civita connection,

$$\mathcal{D}^2 = -g^{\mu\nu} (\nabla_\mu \nabla_\nu) + (\Gamma^\nu - 4T^\nu)(\nabla_\nu + T_\nu) + \frac{1}{2} \gamma^\mu \gamma^\nu [\nabla_\mu + T_\mu, \nabla_\nu + T_\nu]. \tag{197}$$

Here we have used the formulae

$\gamma^\mu [T_\mu, \gamma^\nu] = -4T^\nu$

$\gamma^\mu [\nabla_\mu, \gamma^\nu] = -\gamma^\mu \gamma^\rho \Gamma^\nu_{\mu\rho} = \Gamma^\nu := g^{\mu\rho} \Gamma^\nu_{\mu\rho}$

To simplify the following, we also make use of the fact that the Christoffel symbols and their partial derivatives vanish in Riemann normal coordinates, and $\gamma^{\mu\nu} [\nabla_\mu, \nabla_\nu] = \frac{1}{2} R$, with R the scalar curvature. We can then read off

$$a^\mu = -2(\omega^\mu + 3T^\mu) \tag{198}$$

$$\begin{aligned} b = \frac{1}{2} a^\mu_{,\mu} - \frac{1}{4} a^\mu a_\mu + 5T^\mu_{,\mu} + 2[\omega^\mu, T_\mu] \\ +4T^\mu T_\mu + \tfrac{1}{4} R + \gamma^{\mu\nu} [\nabla_\mu, T_\nu] + \tfrac{1}{2} \gamma^{\mu\nu} [T_\mu, T_\nu]. \end{aligned} \tag{199}$$

As $[\omega^\mu, T_\mu] = g^{\mu\nu} [\omega_\nu, T_\mu] = 0$, and the trace of $T^\mu_{,\mu}$ and vanishes, we have

$$trace_S(b + \frac{1}{4} a^\mu a_\mu - \frac{1}{2} a^\mu_{,\mu}) = 2^{[p/2]} (\frac{1}{4} R - 3 t_{abc} t^{abc}) + trace_S(\gamma^{\mu\nu} [\nabla_\mu, T_\nu]). \tag{200}$$

Here we have used

$trace_S(T^\mu T_\mu) = -\frac{1}{2}t_{abc}t^{abc} \times 2^{[p/2]}$
$trace_S(\frac{1}{2}\gamma^{\mu\nu}[T_\mu, T_\nu]) = -t_{abc}t^{abc} \times 2^{[p/2]}$.
So for the even case we arrive at

$$\begin{aligned} WRes(\mathcal{D}^{-p+2}) = &-\frac{(p-2)Vol(S^{p-1})2^{[p/2]}}{12p(2\pi)^p}\int_X R\sqrt{g}dx \\ &-\frac{(p-2)Vol(S^{p-1})}{2p(2\pi)^p}\int_X (-3t_{abc}t^{abc} + tr(\gamma^{\mu\nu}[\nabla_\mu, T_\nu]))\sqrt{g}dx. \end{aligned} \tag{201}$$

As ∇_μ is torsion free, $\gamma^{\mu\nu}[\nabla_\mu, T_\nu]$ is a boundary term, so

$$WRes(D^{2-p}) = -\frac{(p-2)c(p)}{12}\int_X R\sqrt{g}dx + (p-2)c(p)\int_X \frac{3}{2}t_{abc}t^{abc}dx. \tag{202}$$

This clearly has a unique minimum, given by the vanishing of the torsion term. If we wish to regard the above functional on the affine space of connections, as suggested by Connes, we do the following. Every element of A_σ may be written as $(\mathcal{D}_0 + T) - \mathcal{D}_0$, where $\mathcal{D}_0$ is the Dirac operator of the Levi-Civita connection. Denote this element by T. Then, from what we have proved so far,

$$q(T) := WRes(T^2\mathcal{D}^{-p}) = \frac{(p-2)Vol(S^{p-1})2^{[p/2]}}{p(2\pi)^p}\int_X \frac{3}{2}t_{abc}t^{abc}\sqrt{g}dx. \tag{203}$$

This is clearly a positive definite quadratic form on A_σ, for $p > 2$, and has unique minimum $T = 0$. The value of $WRes(D^{2-p})$ at the minimum is just the other term involving the scalar curvature. Hence, in the even dimensional case, we have completed the proof of the theorem.

4.4.3 The Odd Dimensional Case.

For the odd dimensional case ($p = 2m + 1$) we begin with the observation that $|\mathcal{D}|^{-p+2} = \mathcal{D}^{-2m}|\mathcal{D}|$. As we already know a lot about $\mathcal{D}^{-2}$, the difficult part here will be the absolute value term. So consider the following

$$\begin{aligned} \sigma_{-p}(|\mathcal{D}|^{2-p}) = &\sigma_1(|\mathcal{D}|)\sigma_{-2m-2}(|\mathcal{D}|^{-2m}) + \sigma_0(|\mathcal{D}|)\sigma_{-2m-1}(|\mathcal{D}|^{-2m}) \\ &+\sigma_{-1}(|\mathcal{D}|)\sigma_{-2m}(|\mathcal{D}|^{-2m}) - i\sum_\mu \partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-2m-1}(|\mathcal{D}|^{-2m}) \\ &-i\sum_\mu \partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-2m}(|\mathcal{D}|^{-2m}) \\ &-\frac{1}{2}\sum_{\mu,\nu}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_{-2m}(|\mathcal{D}|^{-2m}). \end{aligned} \tag{204}$$

This tells us that the only terms to compute are $\sigma_1(|\mathcal{D}|)$, $\sigma_0(|\mathcal{D}|)$ and $\sigma_{-1}(|\mathcal{D}|)$, the other terms having been computed earlier. It is a simple matter to convince oneself that $\sigma(|\mathcal{D}|)$ has terms of integral order only by employing

$$\sigma(\mathcal{D}^2) = \sigma(|\mathcal{D}|^2) = \sigma(|\mathcal{D}|)\sigma(|\mathcal{D}|) - i\sum_\mu \partial_{\xi_\mu}\sigma(|\mathcal{D}|)\partial_{x^\mu}\sigma(|\mathcal{D}|) + \text{etc.} \tag{205}$$

Clearly $\sigma_1(|\mathcal{D}|) =\| \xi \|$, which we knew anyway from the multiplicativity of principal symbols. Also

$$
\begin{aligned}
ia^\mu\xi_\mu + b = 2 \parallel \xi \parallel & (\sigma_0(|\mathcal{D}|) + \sigma_{-1}(|\mathcal{D}|) + \sigma_{-2}(|\mathcal{D}|)) \\
& + \sigma_0(|\mathcal{D}|)^2 + 2\sigma_{-1}(|\mathcal{D}|)\sigma_0(|\mathcal{D}|) \\
& - i\sum_\mu \partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_0(|\mathcal{D}|) - i\sum_\mu \partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|) \\
& - i\sum_\mu \partial_{\xi_\mu}\sigma_1(|\mathcal{D}|)\partial_{x^\mu}\sigma_{-1}(|\mathcal{D}|) - i\sum_\mu \partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_0(|\mathcal{D}|) \\
& - i\sum_\mu \partial_{\xi_\mu}\sigma_{-1}(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|) - \frac{1}{2}\sum_{\mu,\nu} \partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|) \\
& - \frac{1}{2}\sum_{\mu,\nu} \partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_0(|\mathcal{D}|) - \frac{1}{2}\sum_{\mu,\nu} \partial^2_{\xi_\mu\xi_\nu}\sigma_0(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|) \\
& + \text{order } -2 \text{ or less.}
\end{aligned} \tag{206}
$$

Looking at the terms of order 1, we have

$$
ia^\mu\xi_\mu = 2 \parallel \xi \parallel \sigma_0(|\mathcal{D}|) - i\sum_\mu \partial_{\xi_\mu} \parallel \xi \parallel \partial_{x^\mu} \parallel \xi \parallel, \tag{207}
$$

or, in Riemann normal coordinates,

$$
\sigma_0(|\mathcal{D}|) =_{RN} \frac{1}{2 \parallel \xi \parallel} ia^\mu\xi_\mu. \tag{208}
$$

The terms of order 0 are more difficult, and we find that

$$
\begin{aligned}
b =_{RN} & \ 2 \parallel \xi \parallel \sigma_{-1}(|\mathcal{D}|) - \frac{1}{4 \parallel \xi \parallel^2} a^\mu\xi_\mu a^\nu\xi_\nu \\
& - i\xi^\mu\partial_{x^\mu}\sigma_0(|\mathcal{D}|) - i\partial_{\xi_\mu}\sigma_0(|\mathcal{D}|)\partial_{x^\mu}\sigma_1(|\mathcal{D}|) \\
& - \frac{1}{2}\partial^2_{\xi_\mu\xi_\nu}\sigma_1(|\mathcal{D}|)\partial^2_{x^\mu x^\nu}\sigma_1(|\mathcal{D}|).
\end{aligned} \tag{209}
$$

Remembering that the derivative of an expression in Riemann normal form is not the Riemann normal form of the derivative, we eventually find that

$$
\begin{aligned}
b =_{RN,\ \mathrm{mod}\ \|\xi\|} & \ 2\sigma_{-1}(|\mathcal{D}|) - \frac{1}{4}a^\mu\xi_\mu a^\nu\xi_\nu \\
& + \frac{1}{2}a^\nu_{,\mu}\xi_\nu\xi^\mu + \frac{1}{12}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.
\end{aligned} \tag{210}
$$

In the above, as well as expressing the result in Riemann normal coordinates and mod $\parallel \xi \parallel$, we have omitted a term proportional to $\xi^\mu\xi^\nu R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma$, since we know that this will vanish when averaged over the cosphere bundle. This gives us an expression for $\sigma_{-1}(|\mathcal{D}|)$ in terms of a^μ and b. Indeed, with the same omissions as above we have

$$
\begin{aligned}
\sigma_{-1}(|\mathcal{D}|) =_{RN,\ \mathrm{mod}\ \|\xi\|} & \ \frac{1}{2}b + \frac{1}{2}a^\mu\xi_\mu a^\nu\xi_\nu \\
& - \frac{1}{8}a^\nu_{,\mu}\xi_\nu\xi^\mu - \frac{1}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.
\end{aligned} \tag{211}
$$

Completing the tedious task of calculation and substitution yields

$$
\begin{aligned}
\sigma_{-p}(|\mathcal{D}|^{2-p}) =_{RN,\ \mathrm{mod}\ \|\xi\|} & -\frac{(p-2)}{2}b + \frac{p(p-2)}{4}a^\nu_{,\mu}\xi_\nu\xi^\mu \\
& - \frac{p(p-2)}{8}a^\mu\xi_\mu a^\nu\xi_\nu + \frac{p(p-2)}{24}\delta^{\mu\nu}R^{\rho\sigma}_{\mu\nu}\xi_\rho\xi_\sigma.
\end{aligned} \tag{212}
$$

We note that the factor $p(p-2)$ arises from $4m^2-1=(2m+1)(2m-1)=p(p-2)$.

Using the experience gained from the even case, we have no trouble integrating this over the cosphere bundle, giving

$$WRes(|\mathcal{D}|^{2-p}) = -\frac{(p-2)}{12}c(p)\int_X R\sqrt{g}d^px + (p-2)c(p)\int_X \frac{3}{2}t_{abc}t^{abc}\sqrt{g}d^px. \tag{213}$$

Again, this expression clearly has a unique minimum (for $p>1$ and odd) given by the Dirac operator of the Levi-Civita connection.

From the results of [24] and the above calculations, if we twist the Dirac operator by some bundle W, the symbol will involve the "twisting curvature" of some connection on W. This does not influence the Wodzicki residue, and so the above result will still hold, except that the minimum is no longer unique. If we have no real structure J, and so are dealing with a spin^c manifold, we have the same value at the minimum, though it is now reached on the linear subspace of self-adjoint $U(1)$ gauge terms. This completes the proof of the theorem.

5 The Abstract Setting

In presenting axioms for noncommutative geometry, Connes has given sufficient conditions for a commutative spectral triple to give rise to a classical geometry, but has not given a simple abstract condition to determine whether an algebra has at least one geometry. In our setup we did not try to remedy this situation, but merely to flesh out some of Connes' ideas enough to give the proof of the above theorem.

In light of this proof, we offer a possible characterisation of the algebras that stand a chance of fulfilling the axioms. The main points are that

1) $\exists c \in Z_n(\mathcal{A}, \mathcal{A}\otimes\mathcal{A}^{op})$ such that $\pi(c)=\Gamma$,

2) $\pi(\Omega^*(\mathcal{A})) \cong \gamma(Cliff(T^*X))$ whilst $\pi(\Omega^*(\mathcal{A}))/\pi(\delta(\ker\pi))$ is the exterior algebra of T^*X.

This second point may be seen as a consequence of the first order condition and the imposition of smoothness on both $\mathcal{A}$ and $\pi(\Omega^*(\mathcal{A}))$. The other interesting feature of the (real) representations of $\Omega^*(\mathcal{A})$, is that if the algebra is noncommutative we have self-adjoint real forms, and so gauge terms. With this in mind we should regard $\Omega^*(\mathcal{A})$, or at least its representations obeying the first order condition, as a generalised Clifford algebra which includes information about the internal (gauge) structure as well. Since this algebra is built on the cotangent space, the following definition is natural.

Definition 5 *A pregeometry is a dense subalgebra $\mathcal{A}$ of a C^*-algebra A such that $\Omega^1(\mathcal{A})$ is finite projective over $\mathcal{A}$.*

This is in part motivated by definitions of smoothness in algebraic geometry, and provides us with our various analytical constraints. Let us explore this.

The hypothesis of finite projectiveness tells us that there exist Hermitian structures on $\Omega^1(\mathcal{A})$. Let us choose one, $(\cdot,\cdot)_{\Omega^1}$. We can then extend it to $\Omega^*(\mathcal{A})$ by requiring homogenous terms of different degree to be orthogonal and

$$(\delta(a)\delta(b),\delta(c)\delta(d))_{\Omega^*} = (\delta(a),\delta(c))_{\Omega^1}(\delta(b),\delta(d))_{\Omega^1}, \tag{214}$$

and so on. Then we can define a norm on $\Omega^*(\mathcal{A})$ by the following equality:

$$\| \delta a \|_{\Omega^*} = \| (\delta a, \delta a) \|_{\mathcal{A}} . \tag{215}$$

As $(\delta a, \delta a)_{\Omega^1} = (\delta a, \delta a)^*_{\Omega^1}$ and $\| \delta a \|_{\Omega^*} = \| (\delta a)^* \|_{\Omega^*}$ we have

$$\| \delta a (\delta a)^* \|_{\Omega^*} = \| \delta a \|^2_{\Omega^*} . \tag{216}$$

So $\Omega^*(\mathcal{A})$ is a normed *-algebra satisfying the C^*-condition, and so we may take the closure to obtain a C^*-algebra.

What are the representations of $\Omega^*(\mathcal{A})$? Let

$$\pi : \Omega^*(\mathcal{A}) \to End(E) \tag{217}$$

be a *-morphism, and E a finite projective module over $\mathcal{A}$. Thus $\pi|_{\mathcal{A}}$ realises E as $E \cong \mathcal{A}^N e$ for some N and some idempotent $e \in M_N(\mathcal{A})$.

As E is finite projective, we have nondegenerate Hermitian forms and connections. Let $(\cdot,\cdot)_E$ be such a form, and ∇_π be a compatible connection. Thus

$$\begin{gathered}\nabla_\pi : E \to \pi(\Omega^1(\mathcal{A})) \otimes E \\ \nabla_\pi(a\xi) = \pi(\delta a) \otimes \xi + a\nabla_\pi \xi \\ (\cdot,\cdot)_E : E \otimes E \to \pi(\mathcal{A}) \\ (\nabla_\pi \xi, \zeta)_E - (\xi, \nabla_\pi \zeta)_E = \pi(\delta(\xi,\zeta)).\end{gathered} \tag{218}$$

If we denote by c the obvious map

$$c : End(E) \otimes E \to E, \tag{219}$$

then we may define

$$\mathcal{D}_\pi = c \circ \nabla_\pi : E \to E. \tag{220}$$

Comparing

$$\mathcal{D}_\pi(a\xi) = \pi(\delta a)\xi + a D_\pi \xi \tag{221}$$

and

$$\mathcal{D}_\pi(a\xi) = [\mathcal{D}_\pi, a]\xi + a D_\pi \xi \tag{222}$$

we see that $\pi(\delta a) = [\mathcal{D}_\pi, \pi(a)]$. Note that $\mathcal{D}_\pi$ depends only on π and the choice of Hermitian structure on $\Omega^1(\mathcal{A})$. This is because all Hermitian metrics on E are equivalent to

$$(\xi, \zeta) = \sum \xi_i \zeta_i^* . \tag{223}$$

This in turn tells us that the definition of compatibility with $(\cdot,\cdot)_E$ reduces to compatibility with the above standard structure. The dependence on the structure on $\Omega^1(\mathcal{A})$ arises from the symmetric part of the multiplication rule on $\Omega^*(\mathcal{A})$ being determined by $(\cdot,\cdot)_{\Omega^1}$. If we are thinking of $(\cdot,\cdot)_\Omega$ as "g" in the differential geometry context, then it is clear that $\mathcal{D}_\pi$ should depend on it if it is to play the role of Dirac operator. Thus it is appropriate to define a representation of $\Omega^*(\mathcal{A})$ as follows.

Definition 6 *Let $\mathcal{A} \subset A$ be a pregeometry. Then a representation of $\Omega^*(\mathcal{A})$ is a *-morphism $\pi : \Omega^*(\mathcal{A}) \to End(E)$, where E is finite projective over $\mathcal{A}$ and such that the first order condition holds.*

In the absence of an operator $\mathcal{D}$, we interpret the first order condition as saying that $\pi(\Omega^0(\mathcal{A}))$ lies in the centre of $\pi(\Omega^*(\mathcal{A}))$, at least in the commutative case. In general, we simply take it to mean that the action of $\pi(\mathcal{A}^{op})$ commutes with the action of $\pi(\Omega^*(\mathcal{A}))$. Next, it is worthwhile pointing out that representations of $\Omega^*(\mathcal{A})$ are a good place to make contact with Connes description of cyclic cohomology via cycles, [6], though this will have to await another occasion. In this definition we encode the first order condition by demanding that $\pi(\Omega^*(\mathcal{A}))$ is a symmetric $\pi(\mathcal{A})$ module in the commutative case. In the noncommutative case that we discuss below, we will require that the image of $\mathcal{A}^{op}$ commutes with the image of $\Omega^*(\mathcal{A})$. Let us consider the problem of encoding Connes' axioms in this setting.

The first thing we require is an extension of these results to $\Omega^*(\mathcal{A}) \otimes \mathcal{A}^{op}$. Since a left module for $\mathcal{A}$ is a right module for $\mathcal{A}^{op}$, we shall have no problem in extending these definitions if we demand that $[\pi(a), \pi(b^{op})] = 0$ for all $a, b \in \mathcal{A}$. Since $\overline{\Omega^*(\mathcal{A})}$ is a C^*-algebra, any representation of it on Hilbert space lies in the bounded operators. This deals with the first two items of Definition 2 in section 3. The real structure will clearly remain as an independent assumption. What remains?

We do not know that $\mathcal{A}$ is "$C^\infty(X)$" in the commutative case yet. Examining the foregoing proof, we see that we first needed to know that the elements involved in the Hochschild cycle c generated $\pi(\mathcal{A})$, which came from $\pi(c) = \Gamma$. Then we needed to show that the condition

$$\pi(a),\ [\mathcal{D}_\pi, \pi(a)] \in \cap^\infty Dom(\delta^m) \tag{224}$$

implied that $\pi(a)$ was a C^∞ function and that $\pi(\Omega^*(\mathcal{A}))$ was the smooth sections of the Clifford bundle. Recall that $\delta(x) = [|\mathcal{D}_\pi|, x]$.

So having a representation, we obtain $\mathcal{D}_\pi$, and we can construct $|\mathcal{D}|_\pi$ if $\mathcal{D}_\pi$ is self-adjoint. This will follow from a short computation using the fact that ∇_π is compatible.

Then we say that π is a smooth representation if

$$\pi(\Omega^*(\mathcal{A})) \subset \cap^\infty Dom(\delta^m). \tag{225}$$

This requires only the finite projectiveness of $\Omega^*(\mathcal{A})$ to state, though this is not necessarily sufficient for it to hold. As $E \cong e\mathcal{A}^N$, this also ensures that $\mathcal{D}_\pi : E \to E$ is well-defined. Further, in the commutative case we see immediately that $\mathcal{D}_\pi$ is an operator of order 1. Thus any pseudodifferential parametrix for $|\mathcal{D}_\pi|$ is an operator of order -1. We can then use Connes' trace theorem to state that $|\mathcal{D}_\pi|^{-p} \in \mathcal{L}^{(1,\infty)}$. The imposition of Poincaré duality then says that $⨍ |\mathcal{D}_\pi|^{-p} \neq 0$.

So, a pregeometry is a choice of "C^1" functions on a space. Given a first order representation π of the universal differential algebra of $\mathcal{A}$ provides an operator $\mathcal{D}_\pi$ of order 1. We use this to impose a further restriction (smoothness) on the representation π and algebra $\mathcal{A}$.

Definition 7 *Let $\pi : \Omega^*(\mathcal{A}) \to End(E)$ be a smooth representation of the pregeometry $\mathcal{A} \subset A$. Then we say that $(\mathcal{A}, \mathcal{D}_\pi, c)$ is a (p, ∞)-summable spectral triple if*

1) $c \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op})$ is a Hochschild cycle with $\pi(c) = \Gamma$

2) Poincaré duality is satisfied

3) E is a pre-Hilbert space with respect to $⨍(\cdot, \cdot)_E |\mathcal{D}|_\pi^{-p}$.

Definition 8 *A real (p, ∞)-summable spectral triple is a (p, ∞)-summable spectral triple with a real structure.*

It is clear that this reformulation loses no information.

This approach may be helpful in relation to the work of [28]. By employing extra operators and imposing supersymmetry relations between them, the authors show that all classical forms of differential geometry (Kähler, hyperkähler, Riemannian...) can also be put into the spectral format. Examining their results show that the converse(s) may also be proved in a similar way to this paper, provided the correct axioms are provided. The elaboration of these axioms may well be aided by the above formulation, but this will have to await another occasion.

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